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1950

CONFERENCE IN TOPOLOGY

Committee

Hassler Whitney (*Chairman*)

Deane Montgomery

N. E. Steenrod

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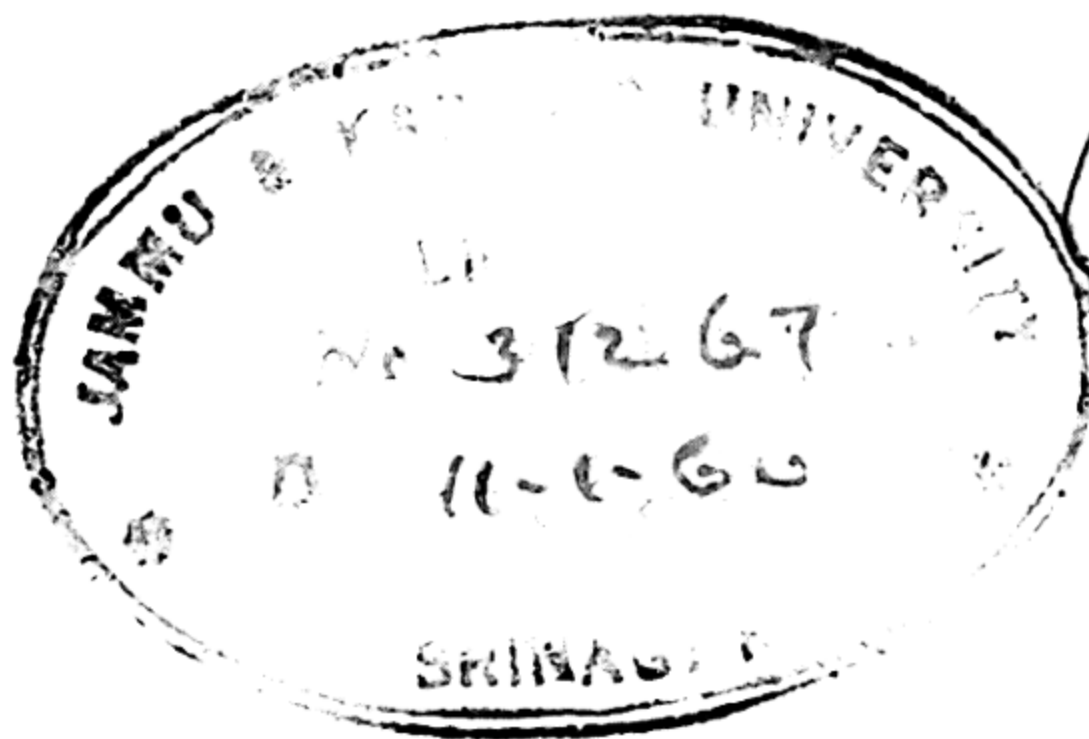
Deane Montgomery

N. E. Steenrod

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Several international conferences in topology have been held. In particular: at Moscow, in 1935 (see *Rec. Math. (Mat. Sbornik)* N.S. vol. 1 (43) No. 5 (1936)); at Geneva, in 1935 (see *l'Enseignement Mathématique* vol. 35 (1936)); at Paris, in 1947 (see *Colloques Internationaux du Centre National de la Recherche Scientifique*, XII, *Topologie Algébrique*, Paris, 1949); and at Brussels, in 1950. In the United States, there have been conferences at the University of Michigan, Ann Arbor, Mich., in 1940 (see *Lectures in Topology*, The University of Michigan Press, 1941), and at Chicago in 1950.

The subject of algebraic topology and applications was chosen for one of the conferences of the Congress because of its great growth in recent years, and the increasingly large contact with other fields of mathematics, in geometry, algebra, and analysis. The subject of general topology has moved considerably into the domain of analysis. It was with great regret that the field of point set theory had to be omitted altogether.

There were four sessions of the conference. The first, on homology and homotopy theory, lies at the foundation of the subject. The second, on fiber bundles and obstructions, presents tools of basic importance for applications in many fields. The third, on differentiable manifolds, is of special importance in analysis, though it relates to other fields as well. The last session, on group theory, lies somewhat apart from the others.

HASSLER WHITNEY

HOMOLOGY AND HOMOTOPY THEORY

HOMOTOPY AND HOMOLOGY

W. HUREWICZ

The concept of homotopy is a mathematical formulation of the intuitive idea of a continuous transition between two geometrical configurations. The concept of homology gives a mathematical precision to the intuitive idea of a curve bounding an "area" or a surface bounding a "volume."

1. The first step toward connecting these two basic concepts of topology was taken by L. E. J. Brouwer in 1912 by demonstrating that two continuous mappings of a two-dimensional sphere into itself can be continuously deformed into each other if and only if they have the same degree (that is, if they are equivalent from the point of view of homology theory).

After having generalized Brouwer's result to an arbitrary number of dimensions, H. Hopf undertook a systematic study of the problem of classifying the continuous mappings of a polytope P into a polytope Q . Each mapping f induces homomorphisms of homology groups of P into the corresponding groups of Q . Two mappings f and g are said to belong to the same *homology class* if they induce identical homomorphisms of homology groups (for all dimensions and all coefficient domains). The mappings f and g are said to belong to the same *homotopy class* if they can be embedded into a common one-parameter continuous family of mappings. The homotopy class of a mapping determines its homology class, but not conversely, as shown by the example of the mappings of the sphere S_3 into S_2 which all belong to the same homology class although there is an infinite number of homotopy classes. The question arises: under what special conditions the homotopy classification of the mappings of P into Q coincides with their homology classification. The classical result of Hopf states that this is the case if P is a polytope of dimension n and Q the n -dimensional sphere S_n . Using cohomology groups instead of homology groups, H. Whitney gave the following elegant formulation to Hopf's theorem. Homotopy classes of mappings of an n -dimensional polytope P into the sphere S_n are in one to one correspondence with the elements of the n -dimensional cohomology group of P with integers as coefficients.

2. In 1934–1935 the author developed the concept and theory of higher dimensional homotopy groups. Given an arcwise connected topological space Y , the n -dimensional homotopy group $\pi_n(Y)$ is defined as follows: Let an arbitrary point $y_0 \in Y$ be singled out once and for all as the "reference point," and let also a fixed point x_0 be selected on the fixed n -sphere S_n . An element of $\pi_n(Y)$ is determined by a continuous mapping of S_n into Y , satisfying the condition $(x_0) = y_0$. Two mappings f and g determine the same element of $\pi_n(Y)$ if and

only if they can be continuously deformed into each other, in such a fashion that the image of x_0 remains at y_0 during the entire process of deformation (this condition can be dispensed with if Y is simply connected). In this case every continuous mapping of S_n into Y determines uniquely an element of $\pi_n(Y)$. The group composition law is defined in a fairly obvious way by identifying two n -spheres tangent at x_0 with the two hemispheres of a single sphere whose equator has been shrunk into x_0 .

An alternative way to introduce homotopy groups is to consider the topological space F of all continuous mappings of S_{n-1} into Y satisfying the condition $f(x_0) = y_0$, where x_0 is a fixed point of S_{n-1} . Although this functional space is, generally speaking, disconnected, it can be shown that its arcwise connected components have the same homotopy type (see below) and consequently have isomorphic fundamental groups. We can thus speak about the fundamental group of the functional space F , and this group turns out to be isomorphic to the n -dimensional homotopy group $\pi_n(Y)$ in the sense of the previous definition. The group $\pi_1(Y)$ is, of course, the fundamental group of Y . A simple geometric argument shows that for $n > 1$ the groups $\pi_n(Y)$ are *abelian*. In contrast with homology groups the homotopy groups of an n -dimensional space may be non-trivial even in dimensions higher than n . For instance, $\pi_3(S_2)$ is an infinite cyclic group (Hopf's theorem) and for $n > 2$ the group $\pi_{n+1}(S_n)$ is of order 2 (Freudenthal-Pontrjagin theorem).

To determine the homotopy groups of a given space is, generally speaking, an extremely difficult problem (even for finite polytopes) which so far has been solved only in a few special cases. In this respect there is a significant difference between homotopy and homology. When a polytope P is broken up in two subpolytopes Q and R , there is a relatively simple relation (Meyer Vietoris theorem restated recently in terms of the so-called exact sequences) between the homology invariants of the polytopes P , Q , R and the intersection $Q \cap R$. No analogous relation exists for homotopy groups. This is tied up to the fact that a continuous image of the n -sphere in P cannot be decomposed into "small" spherical images, the way a simplicial chain can be decomposed into "small parts." Therefore the basic process of homology theory consisting in decomposing a space into smaller pieces with simpler homology structure has no counterpart in homotopy theory. The difficulty is illustrated by the fact that even in the case of a space P represented as the union of two subspaces Q and R with only one point in common, there is no simple relation between higher dimensional homotopy groups of P , Q , and R .

3. In certain "elementary" cases, homotopy groups can be reduced to homology groups. Let Y be an arcwise connected space and let $H_n(Y)$ be the n -dimensional homology group of Y based on singular chains, with integers as coefficients. A continuous image of S_n in Y can be regarded as a singular n -cycle. Since two homotopic spherical images determine homologous singular cycles, one obtains a "natural" homomorphism of $\pi_n(Y)$ into $H_n(Y)$. The fundamental *equivalence theorem* states:

If $n \geq 2$ and the homotopy groups $\pi_i(Y)$ are trivial for $i < n$, the n th homotopy group $\pi_n(Y)$ is isomorphic to the n th homology group $H_n(Y)$ under the natural homomorphism.

For example $\pi_i(S_n)$ is trivial for $i < n$ and hence $\pi_n(S_n)$ is infinite cyclic. Arcwise connected spaces whose homotopy groups in dimensions less than or equal to n vanish are called n -connected. This property is equivalent to the condition that every continuous image of an arbitrary n -dimensional polytope in Y be homotopic to a single point. An obvious corollary to the equivalence theorem states that Y is n -connected if and only if the groups $\pi_1(Y)$, $H_2(Y)$, $\dots$, $H_n(Y)$ are trivial. It follows that a polytope can be shrunk to a point in itself if and only if it is simply connected ($= 1$ -connected) and has vanishing homology groups in all dimensions.

4. The equivalence theorem just stated can be formulated in the following way: If the arcwise connected space Y is $(n-1)$ -connected ($n \geq 2$), the homotopy classes of mappings of S_n into Y coincide with their homology classes. Comparing this result to Hopf's theorem mentioned above we find that the assertions in both theorems are of the same type. Hopf's theorem and the equivalence theorem are both contained in the following more general theorem:

If Y is an $(n-1)$ -connected space ($n \geq 2$) and P an n -dimensional polytope, the homotopy classification of P into Y agrees with their homology classification. More refined results in this direction can be obtained by using the concept of a *homotopy obstruction* developed by S. Eilenberg (implicitly this idea was used for the first time by H. Whitney in his revealing proof of Hopf's theorem). Let Y be a 1-connected space and P an arbitrary polytope. Let us denote by P^m the m -dimensional skeleton of P , that is, the union of all simplexes of P of dimensions less than or equal to m . Consider now two continuous mappings f and g of P into Y . An attempt to deform f continuously into g can be carried out stepwise, each step involving considerations in one dimension only. Suppose we have succeeded in deforming f into a mapping f' which agrees with g on the $(m-1)$ -dimensional skeleton P^{m-1} . For each oriented simplex σ^m of P the images $f'(\sigma^m)$ and $g(\sigma^m)$ (which coincide on the boundary of σ^m) yield, in an obvious fashion, a continuous image of an m -sphere. Let us denote by $\varphi(\sigma^m)$ the element of the homotopy group $\pi_m = \pi_m(Y)$ determined by this spherical image. The function φ can be regarded as an m -dimensional cochain of P with coefficients in the group π_m . This cochain turns out to be a cocycle. Its cohomology class is called the homotopy obstruction for the couple (f', g) . The notation is justified by the following theorem: *If the obstruction is zero (that is, if the cocycle φ is cobounding), the deformation process can be pushed one step further so as to deform f' into a mapping f'' which agrees with g on the m -dimensional skeleton P^m . Moreover the deformation can be carried out in such a way that the image of P^{m-2} (but not necessarily of P^{m-1}) remains unchanged.*

If the cohomology group $H(P, \pi_m)$ of P with coefficients in π_m is trivial (this will be the case, for instance, if $\pi_m = \pi_m(Y)$ vanishes), all obstructions in

dimension m are zero, and hence any two mappings which coincide on P^{m-1} are homotopic on P^m .

The author derived further results by connecting homotopy obstructions with so called *homology obstructions*. Let $H_m = H_m(Y)$ be the m th homology group of Y with integers as coefficients, in the sense of the singular homology theory, and let $H^m(P, H_m)$ be the m th cohomology group of P with coefficients in H_m . The natural homomorphism of π_m into H_m yields a homomorphism of $H(P, \pi_m)$ into $H(P, H_m)$. Under this homomorphism the homotopy obstruction of the couple (f', g) is sent into an element of $H(P, H_m)$ which is called the homology obstruction of the couple (f', g) . The homology obstruction is zero if the mappings f' and g , or—what amounts to the same thing—the mappings f and g belong to the same homology class.

Under certain conditions, homotopy obstructions coincide with homology obstructions. This is, for instance, the case if $\pi_m(Y)$ is isomorphic to $H_m(Y)$ under the natural homomorphism. Under such circumstances the homotopy problem in dimension m is completely reducible to the corresponding homology problem.

5. The groups $\pi_n(Y)$ are a special case of more general invariants called *relative homotopy groups*, which are in many respects analogous to relative homology groups.

Let Y be a topological space and Z a subset of Y . Both Y and Z are assumed to be arcwise connected. For every integer $n \geq 2$ we shall define the relative homotopy group $\pi_n(Y, Z)$. Let E_n be a fixed n -cell with the boundary S_{n-1} . Let us select a point x_0 of S_{n-1} and a point z_0 of Z . An element of $\pi_n(Y, Z)$ is determined by a continuous mapping of E_n into Y satisfying the boundary conditions

$$f(S_{n-1}) \subset Z, \quad f(x_0) = z_0.$$

Two mappings determine the same element of $\pi_n(Y, Z)$ if they can be continuously deformed into each other in such a way that the boundary conditions are satisfied during the entire process of deformation. The composition law is defined by partitioning an n -cell into two cells with an $(n-1)$ -cell in common and shrinking this $(n-1)$ -cell into a single point x_0 . It is evident that no reasonable composition can be defined when $n = 1$. An alternative definition describes relative homotopy groups as fundamental groups of suitably defined functional spaces.

For $n > 2$, $\pi_n(Y, Z)$ is abelian. The group $\pi_2(Y, Z)$ is in general nonabelian, and this accounts for some of the peculiar difficulties encountered in the homotopy theory of two-dimensional spaces.

In exactly the same way as in the case of absolute homotopy groups, one defines a natural homomorphism of the relative homotopy group $\pi_n(Y, Z)$ into the relative homology group $H_n(Y, Z)$ (with integer coefficients). We shall call the couple (Y, Z) *n -connected* ($n \geq 2$) if (a) the group $\pi_1(Z)$ is isomorphic to $\pi_1(Y)$ under the natural homomorphism and (b) $\pi_m(Y, Z)$ vanishes for

$2 \leq m \leq n$. Without using homotopy groups, the definition can be formulated as follows: the couple (Y, Z) is n -connected if given any n -dimensional polytope P with a subpolytope Q and any continuous mapping f of P into Y satisfying $f(Q) \subset Z$, f can be deformed, without changing the image $f(Q)$, into a mapping g satisfying $g(P) \subset Z$. In analogy with the equivalence theorem for absolute homotopy groups we have:

If the couple (Y, Z) is $(n - 1)$ -connected, the homotopy group $\pi_n(Y, Z)$ is isomorphic to the homology group $H_n(Y, Z)$ under the natural homomorphism.

Relative homotopy groups play a basic role in the study of fibre spaces and fibre bundles.

An important generalization of relative homotopy groups has been developed recently by A. L. Blakers and W. S. Massey. They define homotopy groups of a so-called "triad," that is, of a space Y supplied with two closed sets U and V whose union is Y . Roughly speaking, the elements of the n -dimensional homotopy group of a triad are defined by mappings of an n -dimensional cell into Y such that one of the two hemispheres of the boundary of the cell is mapped into U and the other one into V . The theory of homotopy triads helps greatly to understand Freudenthal's so-called "suspension homomorphism" which is the basic tool in the discussion of homotopy groups of spheres.

6. The problem of classifying mappings of one space into another space is closely related to the problem of classifying spaces themselves according to their homotopy properties. Two spaces X and Y are said to have the same *homotopy type* if there exists a continuous mapping f of X into Y and a continuous mapping g of Y into X such that the combined mappings $f \circ g$ and $g \circ f$ are homotopic to identities. Two spaces which have the same homotopy types have isomorphic cohomology rings and isomorphic homotopy groups in all dimensions. As has been shown recently by J. H. C. Whitehead, a necessary and sufficient condition for X and Y to have the same homotopy type is the existence of a continuous mapping f of X into Y which induces isomorphic mappings of the fundamental group and the homology groups of X into the corresponding groups of Y .

J. H. C. Whitehead has succeeded in completely describing the homotopy types of simply connected four-dimensional polytopes in terms of their homology invariants. This description involves in addition to cohomology rings the so-called "Pontrjagin squares."

7. So far we have been concerned mainly with the problem of reducing homotopy properties of mappings and spaces to their homology properties. In certain cases, however, one is led to the converse problem of obtaining information about homology properties of a space from its known homotopy properties. A typical example is an *aspherical space*. By this is meant a space whose homotopy groups vanish in all dimensions $n \geq 2$. It is known that the homotopy type and hence all homology invariants of an aspherical space are determined by its fundamental group. An analogous result holds for spaces which have only one nonvanishing homotopy group. The algebraical process by which in cases of

this type the homology invariants of the space are determined by its homotopy groups has been studied extensively by Eilenberg-MacLane. Their research resulted in a fruitful theory of homology invariants associated with abstract groups. This theory has interesting applications in algebra and in the theory of Lie groups.

8. At present the main effort in homotopy theory seems to concentrate on the problem of determining homotopy groups of spheres. The tools used in this research are predominantly of algebraical nature, like generalized Hopf invariants studied by G. Whitehead, or "cup products" introduced by N. Steenrod. Important advances have been made, most significant of which is the result established recently by G. Whitehead and Pontrjagin, to the effect that $\pi_{n+2}(S_n)$ is a group of order 2 for $n \geq 3$. Nevertheless our knowledge of homotopy groups of spheres remains meager.

Perhaps the present trend of research does not put enough emphasis on tools that could be provided by the geometrical structure from the point of view of differential geometry, like properties of geodesic lines, study of critical points, etc. Recent work of E. Pitcher seems to indicate that some progress can be expected from this direction.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
CAMBRIDGE, MASS., U. S. A.

HOMOTOPY GROUPS AND ALGEBRAIC HOMOLOGY THEORIES

SAMUEL EILENBERG¹

This note will present certain topological results obtained by Saunders MacLane and the author. Some of the algebraic aspects of these questions are presented by MacLane in another note in these Proceedings.

Let X be an arcwise connected topological space with base point x_0 and with vanishing homotopy groups $\pi_i(X)$ for $1 \leq i < n$. The singular homology and cohomology groups of X may then be derived from the singular complex $S_n(X)$ consisting of the singular simplexes whose faces of dimension less than n all degenerate to the point x_0 . A q -simplex T of $S_n(X)$ determines a system of "labels" consisting of elements of $\pi_n = \pi_n(X)$ attached to each n -dimensional face of T . The alternating sum of the labels on the faces of an $(n+1)$ -face of T are zero. Such a system of labels may be regarded as an abstract simplex of a complex $K(\pi_n, n)$; this is a purely algebraic construction on the group $\Pi = \pi_n$ and the integer n . The function which to each simplex of $S_n(X)$ assigns its system of labels yields a simplicial mapping $\kappa: S_n(X) \rightarrow K(\pi_n, n)$. Each n -dimensional simplex of $K(\pi_n, n)$ consists of a single label; i.e., of an element of π_n . This yields the basic cohomology class $b^n \in H^n(\pi_n, n; \pi_n)$ of the complex $K(\pi_n, n)$ with coefficients in π_n , and the basic cohomology class $s^n = \kappa^* b^n \in H^n(X; \pi_n)$ of X with coefficients in π_n .

We shall further assume that the homotopy groups $\pi_i(X)$ vanish also for $n < i < q$. Then every simplex of $K(\pi_n, n)$ of dimension less than or equal to q can be realized geometrically in $S_n(X)$ and this yields an *inverse* simplicial mapping $\bar{\kappa}: K(\pi_n, n) \rightarrow S_n(X)$ defined in dimensions less than or equal to q . Using this map we have shown² that the homology and cohomology groups of X in dimensions less than q (and also partially in dimension q) are those of $K(\pi_n, n)$. In attempting to extend $\bar{\kappa}$ to the dimension $q+1$ one encounters an obstruction which is a cohomology class $k_n^{q+1} \in H^{q+1}(\pi_n, n; \pi_q)$ of the complex $K(\pi_n, n)$ with coefficients in $\pi_q = \pi_q(X)$.

Let K be a (possibly infinite) simplicial complex with ordered vertices and $f: K^n \rightarrow X$ a continuous mapping of the n -skeleton of K . Without loss of generality we may assume that $f(K^{n-1}) = x_0$. If the map f is extendable to a map $K^{n+1} \rightarrow X$, then the cohomology class $f^* s^n \in H^n(K^n; \pi_n)$ determines uniquely a cohomology class $f^\Delta s^n \in H^n(K; \pi_n)$. If further $g: K^n \rightarrow X$ is another such map which agrees with f on a subcomplex L of K , then a relative cohomology class $(f - g)^\Delta s^n \in H^n(K, L; \pi_n)$ is uniquely determined.

Let $f: K^n \cup L \rightarrow X$ be a map extendable to a map $K^{n+1} \cup L \rightarrow X$. For each

¹ John Simon Guggenheim Memorial Fellow.

² S. Eilenberg and S. MacLane, *Relations between homology and homotopy groups of spaces I and II*, Ann. of Math. vol. 46 (1945) pp. 480-509 and vol. 51 (1950) pp. 514-533.

simplex Δ of K and each n -face of Δ , f determines a label which is an element of π_n . These labels form a simplex of $K(\pi_n, n)$, thus yielding a simplicial map $\tilde{f}: K \rightarrow K(\pi_n, n)$. It can also be shown that f is extendable to a map $f': K^q \cup L \rightarrow X$. The obstruction $c^{q+1}(f') \in H^{q+1}(K, L; \pi_q)$ is independent of the choice of f' and is called the secondary obstruction $z^{q+1}(f)$ of f .

THEOREM 1. *Let K be a simplicial complex, L a subcomplex, $f, g: K^n \cup L \rightarrow X$ maps such that $f(K^{n-1}) = g(K^{n-1}) = x_0$ and $f|L = g|L$. If both f and g are extendable to maps $K^{n+1} \cup L \rightarrow X$, then their secondary obstructions satisfy*

$$z^{q+1}(f) - z^{q+1}(g) = (\tilde{f} - \tilde{g}) * k_n^{q+1}.$$

Here $(\tilde{f} - \tilde{g})^*$ is the difference homomorphism $H^{q+1}(\pi_n, n; \pi_q) \rightarrow H^{q+1}(K, L, \pi_q)$ induced by the maps $\tilde{f}$ and $\tilde{g}$ which agree on L .

The above theorem yields interesting results only if one has more information about the element k_n^{q+1} and about the group $H^{q+1}(\pi_n, n; \pi_q)$ in which it lies. The latter is closely related with the homology theory of abelian groups based on the complexes $A(\Pi)$ and $A^1(\Pi)$ of the aforementioned note of MacLane.

We limit our attention here to the case $q = n + 1$. In this case we have natural isomorphisms

$$\begin{aligned} H^4(\Pi, n; G) &\approx H^3(A^1(\Pi); G) \\ H^{n+2}(\Pi, n; G) &\approx H^3(A(\Pi); G), \end{aligned} \quad n > 2.$$

Each element χ of $H^{n+2}(\Pi, n; g)$ yields a trace t which is a function defined on Π with values in G and which satisfies the conditions

- (1) $t(x) - t(-x) = 0$
 - (2) $t(x + y + z) - t(y + z) - t(x + z) - t(x + y) + t(x) + t(y) + t(z) = 0$
- for $n = 2$. For $n > 2$, (2) is replaced by the stronger condition

$$(2') \quad t(x + y) - t(y) - t(x) = 0.$$

In particular, we consider the element $k_n^{n+2} \in H^{n+2}(\pi_n, n; \pi_{n+1})$, and prove that the trace of k_n^{n+2} is J. H. C. Whitehead's function $\eta: \pi_n \rightarrow \pi_{n+1}$ obtained by combining each map $S^n \rightarrow X$ with a map $S^{n+1} \rightarrow S^n$ that yields a generator of $\pi_{n+1}(S^n)$.³

The correspondence $\chi \rightarrow t$ yields an isomorphic mapping of the group $H^{n+2}(\Pi, n; G)$ onto the group of all functions satisfying conditions (1) and (2) (or (1) and (2') if $n > 2$). If the abelian group Π is finitely generated, then the inverse mapping may be described as follows.

Let K be a complex with sufficient simplicial structure, L a subcomplex, and let $t: \Pi \rightarrow G$ be a function satisfying (1) and (2). Then if Π is finitely generated, one may define a "Pontrjagin square"

$$P_t: H^q(K, L; \Pi) \rightarrow H^{2q}(K, L; G), \quad q \text{ even},$$

³ Cf. G. W. Whitehead, *On spaces with vanishing low-dimensional homotopy groups*, Proc. Nat. Acad. Sci. U. S. A. vol. 34 (1948) pp. 207-211; Theorem 5.

which itself satisfies (1) and (2). If t satisfies (1) and (2'), then one has a "Steenrod square"

$$Sq_i^2: H^q(K, L; \Pi) \rightarrow H^{q+2}(K, L; G)$$

which itself satisfies (1) and (2'). If $b^n \in H^n(\Pi, n; \Pi)$ is the basic cohomology class of $K(\Pi, n)$, then the elements $P_i(b^2)$ and $Sq_i^2(b^n)$ for $n > 2$ of $H^{n+2}(\Pi, n; G)$ have precisely the trace t .

These results combined with some formal properties of P_i and Sq_i^2 and with Theorem 1 yield the following theorem.

THEOREM 2. *Let X be an arcwise connected topological space with $\pi_i(X) = 0$ for $i < n$ ($n > 1$) and with $\pi_n(X)$ finitely generated. Let K be a simplicial complex, L a subcomplex, and $f, g: K^n \cup L \rightarrow X$ two maps extendable to maps $K^{n+1} \cup L \rightarrow X$ which agree on L . Then their secondary obstructions satisfy*

$$z^{n+2}(f) - z^{n+2}(g) = \begin{cases} P_\eta(\lambda^2) + g^\Delta(s^2) \cup \lambda^2, & n = 2, \\ f^\Delta(s^2) \cup \lambda^2 - P_\eta(\lambda^2), & n = 2, \\ Sq_\eta^2(\lambda^n), & n > 2, \end{cases}$$

where $s^n \in H^n(X; \pi_n)$ is the basic cohomology class of X , $\lambda^n = (f - g)^\Delta s^n$, and $\cup$ denotes the ordinary cup product relative to the pairing $[x, y] = \eta(x + y) - \eta(x) - \eta(y)$.

In order to deduce from this theorem a classification theorem for maps $K^{n+1} \rightarrow X$ we need the "Postnikov square" which is a homomorphism

$$\bar{P}_i: H^q(K, L; \Pi) \rightarrow H^{2q+1}(K, L; G), \quad q \text{ odd},$$

defined for each t satisfying conditions (1) and (2).

THEOREM 3. *Let X be as in Theorem 2 and let $f, g: K^{n+1} \rightarrow X$ be two maps which agree on $K^n \cup L$. Let $d^{n+1}(f, g) \in H^{n+1}(K, L; \pi_{n+1})$ be the cohomology class measuring the difference between f and g . Then f and g are homotopic relative to L if and only if there is a cohomology class $e^{n-1} \in H^{n-1}(K, L; \pi_n)$ such that*

$$d^{n+1}(f, g) = \begin{cases} \bar{P}_\eta(e^1) + f^*(s^2) \cup e^1, & n = 2, \\ Sq_\eta^2(e^{n-1}), & n > 2. \end{cases}$$

Theorems 2 and 3 constitute a generalization of Steenrod's results⁴ for the case $X = S^n$. It should be noted that the method also is in a sense a generalization

⁴ N. E. Steenrod, *Products of cocycles and extensions of mappings*, Ann. of Math. vol. 48 (1947) pp. 290-320.

of that of Steenrod: the complexes M^n are replaced here by the algebraic complexes $K(\pi_n, n)$. Our results are almost identical with results recently obtained by J. H. C. Whitehead by a different method and include earlier results of Whitney⁵ and Postnikov.⁶

COLUMBIA UNIVERSITY,
NEW YORK, N. Y., U. S. A.

⁵ H. Whitney, *Classification of the mappings of a 3-complex into a simply connected space*, Ann. of Math. vol. 50 (1949) pp. 270–284.

⁶ M. M. Postnikov, *Classification of the continuous mappings of an arbitrary n -dimensional polyhedron into a connected topological space which is aspherical in dimensions greater than unity and less than n* , C. R. (Doklady) Acad. Sci. URSS N.S. vol. 67 (1949) pp. 427–430

ALGEBRAIC HOMOTOPY THEORY

J. H. C. WHITEHEAD

In homotopy theory, spaces are classified in terms of homotopy classes of maps, rather than individual maps of one space in another. Thus, using the word category in the sense of S. Eilenberg and Saunders MacLane [1],¹ a homotopy category of spaces is one in which the objects are topological spaces and the "mappings" are not individual maps but homotopy classes of ordinary maps. The equivalences are the classes with two-sided inverses, and two spaces are of the same homotopy type if and only if they are related by such an equivalence. The ultimate object of algebraic homotopy is to construct a purely algebraic theory, which is equivalent to homotopy theory in the same sort of way that "analytic" is equivalent to "pure" projective geometry. In discussing this project I shall confine myself to spaces which are covered by CW-complexes as defined in [4].

The following theorem is proved in [4]. Let π_n, π'_n be the n th homotopy groups of K, K' . Let $\phi: K \rightarrow K'$ be a map of a complex K into a complex K' , which induces an isomorphism $\phi_n: \pi_n \approx \pi'_n$ for every $n = 1, 2, \dots$. Then $\phi: K \equiv K'$, which means that the homotopy class containing ϕ is an equivalence. Thus the homotopy groups constitute a system of algebraic invariants which, in a certain sense, are sufficiently powerful to characterize the homotopy type of a complex.

This does not mean that $K \equiv K'$ just because $\pi_n \approx \pi'_n$ for every $n = 1, 2, \dots$. The crux of the matter is not merely that $\pi_n \approx \pi'_n$, but that a certain family of isomorphisms, $\phi_n: \pi_n \approx \pi'_n$, has a *geometrical realization*, $K \rightarrow K'$. That is to say, the latter induces the former. Therefore the emphasis is shifted to the realization problem, which is to find necessary and sufficient conditions in order that a given set of isomorphisms or, more generally, homomorphisms, $\phi_n: \pi_n \rightarrow \pi'_n$, have a geometrical realization $K \rightarrow K'$.

At this stage let me remark, in parentheses, that other sets of "sufficiently powerful" invariants may be defined in terms of the universal covering complex, $\tilde{K}$, of K . For the two preceding paragraphs may be restated in terms of the system of groups $\pi_1, H_n(\tilde{K})$, where $H_n(\tilde{K})$ is the n th integral homology group, defined in terms of finite chains of $\tilde{K}$. Thus, within the category of simply connected complexes, π_n could be replaced by $H_n(K)$. However I shall continue the discussion in terms of the groups π_n .

When studying a complex K it is natural to consider in succession the sections $K^1, K^2, \dots$, where K^n consists of all the cells in K of at most n dimensions. Now the homotopy type of K^n is not an invariant of K . Therefore consider the n -type, this being a homotopy invariant of K , which depends only on K^n . Two complexes K, K' are of the same n -type if and only if there are maps $\phi: K^n \rightarrow K'^n$,

¹ Numbers in brackets refer to the list of references at the end of this paper.

$\phi': K'^n \rightarrow K^n$ such that

$$\phi'\phi \mid K^{n-1} \simeq 1, \quad \phi\phi' \mid K'^{n-1} \simeq 1,$$

where $\simeq$ indicates homotopy in K^n , K'^n and each 1 stands for the appropriate identical map $K^{n-1} \rightarrow K^n$ or $K'^{n-1} \rightarrow K'^n$. Subject to this condition we write $\phi: K^n \equiv_{n-1} K'^n$. It is the fact that the above homotopies are in K^n , K'^n , not in K^{n-1} , K'^{n-1} , which makes the n -type a homotopy invariant. It is proved in [4], assuming K, K' to be connected, that $\phi: K^n \equiv_{n-1} K'^n$ if and only if $\phi_r: \pi_r \approx \pi'_r$ for $r = 1, \dots, n-1$, where ϕ_r is induced by ϕ . Hence it follows that K, K' are of the same n -type if they are of the same m -type for any $m > n$, where $m \leq \infty$ and the ∞ -type means the homotopy type. Therefore the n -type is a homotopy invariant and a fortiori a topological invariant of the space covered by K . Within the category of at most $(n-1)$ -dimensional complexes the n -type is the same as the homotopy type.

All connected complexes are of the same 1-type, so that the interest begins with $n = 2$. Any homomorphism $\pi_1 \rightarrow \pi'_1$ has a geometrical realization $K^2 \rightarrow K'^2$. Therefore it follows from the theorem quoted in the preceding paragraph (or from an easy ad hoc argument) that two complexes K, K' are of the same 2-type if and only if $\pi_1 \approx \pi'_1$. Moreover, given a group G , there is a complex K such that $\pi_1 \approx G$. Thus π_1 is an "algebraic equivalent" of the 2-type. It is natural to consider the problem of finding an algebraic equivalent, T^n , of the n -type. I shall explain what has been done in case $n = 3$ or, for simply connected complexes, in case $n = 4$. This will indicate what I mean by an algebraic equivalent of the n -type.

Let Q be an arbitrary (multiplicative) group and G an arbitrary (additive) group. Let $\eta: Q \rightarrow A(G)$ be a homomorphism of Q into the group, $A(G)$, of automorphisms of G , by means of which Q is expressed as a group of operators on G . Using these operators we define the n th cohomology group, $H^n(Q, G)$, of Q , with coefficients in G . Then an algebraic 3-type is defined as a quadruple, $T^3 = (Q, G, \eta, \mathbf{k})$, where $\mathbf{k}$ is an arbitrary element of $H^3(Q, G)$. Let $T'^3 = (Q', G', \eta', \mathbf{k}')$ be any algebraic 3-type. Then a homomorphism $F: T^3 \rightarrow T'^3$ consists of a pair of homomorphisms $f: Q \rightarrow Q'$, $h: G \rightarrow G'$ such that

$$\begin{aligned} h\eta(q) &= \eta'(fq)h: G \rightarrow G' \\ f^*\mathbf{k}' &= h_*\mathbf{k} \in H^3(Q, G'), \end{aligned}$$

where $f^*: H^3(Q', G') \rightarrow H^3(Q, G')$ and $h_*: H^3(Q, G) \rightarrow H^3(Q, G')$ are induced by f, h . If f, h are isomorphisms (onto) we write $F: T^3 \approx T'^3$. It is proved in [3] that:

(1) Any complex, K , determines an algebraic 3-type, $T^3(K)$, in which $Q = \pi_1$, $G = \pi_2$, η is the homomorphism determined by the way in which π_1 operates, as usual, on π_2 and $\mathbf{k}$ is the Eilenberg-MacLane invariant (see [2] or [3]).

(2) The homomorphisms $f: \pi_1 \rightarrow \pi'_1$, $h: \pi_2 \rightarrow \pi'_2$, which are induced by a map $K \rightarrow K'$, satisfy the above conditions for a homomorphism $T^3(K) \rightarrow T^3(K')$.

(3) Any homomorphism $T^3(K) \rightarrow T^3(K')$ has a geometrical realization $K \rightarrow K'$.

(4) If T^3 is given, there is a complex, K , such that $T^3 \approx T^3(K)$.

In consequence of these conditions an algebraic 3-type may be described as an algebraic equivalent of the 3-type of a complex.

Now consider the category of simply connected complexes. Given any (additive) Abelian group, A , we define a group $\Gamma(A)$ as follows. It has a set of generators $\gamma(a)$, which is indexed to A , and these are subject to the (complete) set of relations:

$$\gamma(-a) = \gamma(a)$$

$$\gamma(a + b + c) - \gamma(b + c) - \gamma(c + a) - \gamma(a + b)$$

$$+ \gamma(a) + \gamma(b) + \gamma(c) = 0.$$

A homomorphism $f: A \rightarrow A'$, where A' is Abelian, induces a homomorphism $\Gamma f: \Gamma(A) \rightarrow \Gamma(A')$, which is given by $\Gamma f \gamma(a) = \gamma(fa)$. Then a (simply connected) algebraic 4-type is a triple $T_0^4 = (A, B, i)$, where A, B are arbitrary Abelian groups and $i: \Gamma(A) \rightarrow B$ is an arbitrary homomorphism. A homomorphism $F: T_0^4 \rightarrow T_0'^4$, where $T_0'^4 = (A', B', i')$ consists of a pair of homomorphisms $f: A \rightarrow A', g: B \rightarrow B'$, such that the diagram

$$\begin{array}{ccc} \Gamma(A) & \xrightarrow{i} & B \\ \downarrow \Gamma f & & \downarrow g \\ \Gamma(A') & \xrightarrow{i'} & B' \end{array}$$

is commutative.

Let K be a simply connected complex and let $\Gamma_3(K)$ be the image of $\pi_3(K^2)$ in the injection $\pi_3(K^2) \rightarrow \pi_3(K^3)$. Though apparently combinatorial in character, the group $\Gamma_3(K)$ is a topological invariant of K . In fact it is an invariant of the 3-type of K . Moreover it is the image of $\Gamma(\pi_2)$ in a natural isomorphism $\theta: \Gamma(\pi_2) \approx \Gamma_3(K)$. Let $i': \pi_3(K^3) \rightarrow \pi_3$ be the injection, let $i': \Gamma_3(K) \rightarrow \pi_3$ be the homomorphism determined by i' and let

$$i = i'\theta: \Gamma(\pi_2) \rightarrow \pi_3.$$

It is proved in [5] that, within the category of simply connected complexes, $T_0^4 = (\pi_2, \pi_3, i)$ is an algebraic equivalent of the 4-type of K .

It should be stated that, in these cases, the relation between algebra and geometry is incomplete. For it cannot be stated, as it can in case $n = 2$, that maps $\phi_0, \phi_1: K^n \rightarrow K'^n$ ($n = 3, 4$), which induce the same homomorphism $T^n(K) \rightarrow T^n(K')$, are $(n - 1)$ -homomorphic, meaning that $\phi_0|K^{n-1} \simeq \phi_1|K^{n-1}$ in K'^n .

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OXFORD UNIVERSITY,
OXFORD, ENGLAND.

HOMOTOPY GROUPS OF SPHERES

GEORGE W. WHITEHEAD

The homotopy groups of a topological space, discovered by Hurewicz¹ in 1935, have become an important tool in studying the classification and extension problems. The homotopy groups of a space have many properties in common with the homology groups; an important difference is that the homotopy groups do not have the excision property.

A special case of the excision problem is the following. Let S^n be an n -sphere, and let S^{n-1} be an equator dividing S^n into two "hemispheres", E_+^n and E_-^n . Then the excision homomorphism maps $\pi_i(E_+^n, S^{n-1})$ into $\pi_i(S^n, E_-^n)$. Now the boundary operator maps $\pi_i(E_+^n, S^{n-1})$ isomorphically onto $\pi_{i-1}(S^{n-1})$, while $\pi_i(S^n)$ is mapped isomorphically onto $\pi_i(S^n, E_-^n)$ by the inclusion map. Hence the excision homomorphism induces a homomorphism $E: \pi_{i-1}(S^{n-1}) \rightarrow \pi_i(S^n)$. This homomorphism is the Freudenthal *suspension*.² This suspension operator has proved to be an important tool in studying $\pi_i(S^n)$.

The groups $\pi_i(S^n)$ have long been known for $i \leq n$; for $i < n$, $\pi_i(S^n) = 0$, while $\pi_n(S^n)$ is infinite cyclic. It is also known that $\pi_i(S^1) = 0$ for $i > 1$.

The first attack on $\pi_i(S^n)$ for $i > n$ was made by Hopf,³ who defined a homomorphism H_0 of $\pi_{2n-1}(S^n)$ into the additive group of integers and proved that $\pi_{2n-1}(S^n)$ is not zero, and in fact contains an element of infinite order, provided that n is even. It was further shown by Hurewicz¹ that $\pi_3(S^2)$ is infinite cyclic, and in fact the homomorphism H_0 is an isomorphism of $\pi_3(S^2)$ onto the group of integers. Hopf proved that if n is odd, then the *Hopf invariant* $H_0(\alpha)$ is zero for every $\alpha \in \pi_{2n-1}(S^n)$; while if n is even, there exists $\alpha \in \pi_{2n-1}(S^n)$ with $H_0(\alpha) = 2$. If $n = 2, 4$, or 8 , there is an element of $\pi_{2n-1}(S^n)$ with Hopf invariant 1. Hopf raised the question of whether $\pi_{2n-1}(S^n)$ contains an element of Hopf invariant 1 for other values of n . This question was partially settled by the author,⁴ who showed that no such element exists if $n \equiv 2 \pmod{4}$ and $n > 2$.

J. H. C. Whitehead⁵ has defined an operation of multiplication between the homotopy groups of a topological space X . This operation associates with each $\alpha \in \pi_p(X)$, $\beta \in \pi_q(X)$, an element $[\alpha, \beta] \in \pi_{p+q-1}(X)$. This operation is bilinear if $p > 1$ and $q > 1$; while if p or $q = 1$ it can be expressed in terms of the operations of the fundamental group on the higher homotopy groups.⁶ The bracket product has the following additional properties:

¹ W. Hurewicz, Proceedings of the K. Akademie van Wetenschappen Amsterdam vol. 38 (1935) pp. 112-119, 521-528; vol. 39 (1936) pp. 117-126, 215-224.

² H. Freudenthal, Compositio Math. vol. 5 (1937) pp. 299-314.

³ H. Hopf, Math. Ann. vol. 104 (1931) pp. 637-665; Fund. Math. vol. 25 (1935) pp. 427-440.

⁴ G. W. Whitehead, Ann. of Math. vol. 51 (1950) pp. 192-237.

⁵ J. H. C. Whitehead, Ann. of Math. vol. 42 (1941) pp. 409-428.

⁶ S. Eilenberg, Fund. Math. vol. 32 (1939) pp. 167-175.

- 1) $[\beta, \alpha] = (-1)^{pq}[\alpha, \beta]$;
- 2) if ι_r is the element of $\pi_r(S^r)$ represented by the identity map, then

$$H_0([\iota_r, \iota_r]) = \begin{cases} 2 & (r \text{ even}) \\ 0 & (r \text{ odd}). \end{cases}$$

The groups $\pi_i(S^n)$ with $n < i < 2n - 1$ were studied by Freudenthal.² The Freudenthal theorems, with some improvements due to the author,⁴ can be stated as follows:

- (1) $E: \pi_i(S^n) \rightarrow \pi_{i+1}(S^{n+1})$ is onto if $i \leq 2n - 1$;
- (2) $E: \pi_i(S^n) \rightarrow \pi_{i+1}(S^{n+1})$ is an isomorphism if $i < 2n - 1$;
- (3) the image of $E: \pi_{2n-2}(S^{n-1}) \rightarrow \pi_{2n-1}(S^n)$ is the kernel of H_0 ;
- (4) the kernel K of $E: \pi_{2n-1}(S^n) \rightarrow \pi_{2n}(S^{n+1})$ is the subgroup generated by $[\iota_n, \iota_n]$; (a) if n is even, K is infinite cyclic; (b) if n is odd, K is zero or cyclic of order two, according as $\pi_{2n+1}(S^{n+1})$ contains an element of Hopf invariant 1 or not.

Any generalization of (3) would seem to require a generalization of the Hopf homomorphism H_0 . A partial generalization of H_0 was introduced by the author,⁴ who defined a homomorphism $H: \pi_n(S^r) \rightarrow \pi_n(S^{2r-1})$ for each $n < 3r - 3$ as follows.

Denote by $S^r \vee S^r$ the subset of $S^r \times y_0 \cup y_0 \times S^r$ of $S^r \times S^r$; $S^r \vee S^r$ is the union of two r -spheres with a point in common. Let $\phi: S^r \rightarrow S^r \vee S^r$ be a map which pinches S^{r-1} to the point $y_0 \times y_0$ and maps each open hemisphere topologically. Then ϕ induces a homomorphism $\phi_*: \pi_n(S^r) \rightarrow \pi_n(S^r \vee S^r)$.

In the homotopy sequence of the pair $(S^r \times S^r, S^r \vee S^r)$, the boundary homomorphism $\partial: \pi_{n+1}(S^r \times S^r, S^r \vee S^r) \rightarrow \pi_n(S^r \vee S^r)$ is an isomorphism into and the inclusion homomorphism $i_*: \pi_n(S^r \vee S^r) \rightarrow \pi_n(S^r \times S^r)$ is onto; moreover, $\pi_n(S^r \vee S^r)$ is in a natural way the direct sum of Image ∂ with a subgroup mapped isomorphically by i_* . Let $Q: \pi_n(S^r \vee S^r) \rightarrow \text{Image } \partial$ be the projection associated with this direct sum decomposition.

Let j', j'' be the inclusion maps of $S^r \times y_0$ and $y_0 \times S^r$ into $S^r \vee S^r$. Then j'_* and j''_* are isomorphisms into, and $j'_*\pi_r(S^r \times y_0)$ and $j''*\pi_r(y_0 \times S^r)$ are infinite cyclic groups generated by ι'_r, ι''_r .

It turns out that if $f: S^{2r-1} \rightarrow S^r \vee S^r$ is a map representing $[\iota'_r, \iota''_r]$, then f_* maps $\pi_n(S^{2r-1})$ isomorphically onto Image ∂ for $n \leq 3r - 3$. We then define $H: \pi_n(S^r) \rightarrow \pi_n(S^{2r-1})$ by

$$H = f_*^{-1} \partial^{-1} Q \phi_*.$$

If $n = 2r - 1$, then H maps $\pi_{2r-1}(S^r)$ into the infinite cyclic group $\pi_{2r-1}(S^{2r-1})$ and $H(\alpha) = H_0(\alpha) \iota_{2r-1}$. Thus H is a generalization of H_0 . The definition of H has been improved and extended by Blakers and Massey⁷ and by Hilton.⁸

⁷ A. L. Blakers and W. S. Massey, Proc. Nat. Acad. Sci. U. S. A. vol. 35 (1949) pp. 322-328. A full account will appear in Ann. of Math.

⁸ Unpublished.

The attempt to generalize Freudenthal's theorems, based on the definition of H , has met with only partial success. It turns out that $H \cdot E = 0$, so that $\text{Kernel } H \supset \text{Image } E$, but it is not known whether the opposite inclusion is true. Only a partial generalization of (4) has been obtained: if $\alpha \in \pi_n(S^r)$ and $E\alpha = 0$, then $H(\alpha) = 0$ if r is odd and $H(\alpha) \in 2\pi_n(S^{2r-1})$ if r is even. These facts are used by the author in the partial solution to Hopf's problem mentioned earlier, as well as in the proof⁹ of the fact that $\pi_{n+2}(S^n)$ is of order two for $n \geq 2$. (The latter theorem was proved independently by Pontrjagin¹⁰ by entirely different methods.)

Another operation involving homotopy groups of spheres is that of composition. If X is a topological space, and if $f: S^n \rightarrow S^r$, $g: S^r \rightarrow X$ are maps, then $g \circ f: S^n \rightarrow X$, and the operation $(f, g) \rightarrow g \circ f$ induces an operation on homotopy classes, associating with each $\alpha \in \pi_n(S^r)$, $\beta \in \pi_r(X)$ an element $\beta \circ \alpha \in \pi_n(X)$. For this operation the left distributive law

$$\beta \circ (\alpha_1 + \alpha_2) = \beta \circ \alpha_1 + \beta \circ \alpha_2$$

is valid, but right distributivity does not in general hold. The deviation from right distributivity is expressed by the formula⁴

$$(\beta_1 + \beta_2) \circ \alpha = \beta_1 \circ \alpha + \beta_2 \circ \alpha + [\beta_1, \beta_2] \circ H(\alpha)$$

provided that $n < 3r - 2$. From this we can deduce the following identities:

$$(\beta_1 + \beta_2) \circ \alpha - \beta_1 \circ \alpha - \beta_2 \circ \alpha = 0 \quad \text{if } n < 2r - 1;$$

$$\begin{aligned} &(\beta_1 + \beta_2 + \beta_3) \circ \alpha - (\beta_1 + \beta_2) \circ \alpha - (\beta_1 + \beta_3) \circ \alpha \\ &\quad - (\beta_2 + \beta_3) \circ \alpha + \beta_1 \circ \alpha + \beta_2 \circ \alpha + \beta_3 \circ \alpha = 0 \quad \text{if } n < 3r - 2. \end{aligned}$$

It is not known whether the identities suggested by these formulas hold if $n \geq 3r - 2$.

The homotopy groups of a space X were originally defined by Hurewicz¹ as the fundamental groups of certain function spaces over X . Let $y_0 \in S^n$ and let $F^n(X, x_0)$ be the space of all maps of (S^n, y_0) into (X, x_0) , topologized by the compact-open topology. Let $\alpha \in \pi_n(X)$ and let $F_\alpha^n(X, x_0)$ be the path-component of $F^n(X, x_0)$ consisting of all maps which represent the element $\alpha \in \pi_n(X)$. Hurewicz proved that all the spaces $F_\alpha^n(X, x_0)$ have the same homotopy type, and that $\pi_k(F^n(X, x_0)) \approx \pi_{n+k}(X)$. One may therefore attempt to study the homotopy groups of X by examining the function spaces $F_\alpha^n(X, x_0)$.

The case $n = 1$, $\alpha = 0$, $X = S^p$ has recently been studied by Pitcher.¹¹ The space $F_0^1(S^p, x_0)$ is the space of closed paths in S^p starting and ending at x_0 ; this space has the same homotopy type as the space of paths joining two fixed

⁹ G. W. Whitehead, Ann. of Math. vol. 52 (1950) pp. 245-247.

¹⁰ L. Pontrjagin, C. R. Acad. Sci. URSS vol. 70 (1950) pp. 957-959.

¹¹ E. Pitcher, Bull. Amer. Math. Soc. Abstract 56-1-38.

points of S^p . Pitcher attacks the problem by means of the powerful tools of critical point theory in the calculus of variations.

Suppose that $n = p$, $\alpha = \iota_p$. Then the rotation group R_{p-1} of S^{p-1} is naturally imbedded in $F_{\iota_p}^p(S^p, x_0)$. Thus there is a natural homomorphism $J: \pi_q(R_{p-1}) \rightarrow \pi_{p+q}(S^p)$. This homomorphism was used by Hopf³ in his study of $\pi_{2p-1}(S^p)$. The homomorphism J turns out to be an isomorphism onto if $q = 1$, but for $q = 2$, J is no longer onto since $\pi_2(R_{p-1}) = 0$ and $\pi_{p+2}(S^p) \neq 0$. However, the nonzero element of $\pi_{p+2}(S^p)$ can be obtained from the rotation group in another way.

Since R_{p-1} is a topological group, the set of homotopy classes of maps of $S^q \times S^r$ into R_{p-1} forms in a natural way a group $\pi_{q,r}(R_{p-1})$. There is a map J' of $\pi_{q,r}(R_{p-1})$ into $\pi_{p+q+r}(S^p)$, analogous to J , defined as follows. Let S^{p+q+r} be referred to coordinates (u, v, w) where $u \in E^{p-1}$, $v \in E^{q+1}$, $w \in E^{r+1}$ and $|u|^2 + |v|^2 + |w|^2 = 1$. The subset C of S^{p+q+r} defined by the equation

$$|v|^2 + |w|^2 = 8|v|^2|w|^2$$

is a homeomorphic copy of $S^{p-1} \times S^q \times S^r$, separating S^{p+q+r} into two subsets A, B . If $f: S^q \times S^r \rightarrow R_{p-1}$, then f defines a map $f': S^{p-1} \times S^q \times S^r \rightarrow S^{p-1}$ by

$$f'(x, y, z) = f(y, z)(x).$$

Then f' may be considered as a map of C into S^{p-1} , and can be extended to a map $f'': S^{p+q+r} \rightarrow S^r$ such that $f''(\bar{A}) \subset E_+^r$, $f''(\bar{B}) \subset E_-^r$. If f is homotopic to f_1 , then f'' is homotopic to f_1'' , and therefore the correspondence $f \rightarrow f''$ induces a map $J': \pi_{q,r}(R_{p-1}) \rightarrow \pi_{p+q+r}(S^p)$.

The group $\pi_{q,r}(R_{p-1})$ has the following structure. The set of homotopy classes of maps of the form $f(x, y) = g(x)$, where $g: S^q \rightarrow R_{p-1}$, forms a group isomorphic with $\pi_q(R_{p-1})$. Similarly, $\pi_{q,r}(R_{p-1})$ contains an isomorphic copy of $\pi_r(R_{p-1})$. The set of homotopy classes of maps of the form $f: (S^q \times S^r, S^q \wedge S^r) \rightarrow (R_{p-1}, r_0)$ forms a third subgroup isomorphic with $\pi_{q+r}(R_{p-1})$. Then each element of $\pi_{q+r}(R_{p-1})$ is uniquely expressible in the form $\alpha\beta\gamma$, where $\alpha \in \pi_q(R_{p-1})$, $\beta \in \pi_r(R_{p-1})$, and $\gamma \in \pi_{q+r}(R_{p-1})$; the subgroup $\pi_{q+r}(R_{p-1})$ is contained in the center of $\pi_{q,r}(R_{p-1})$ and the factor group is isomorphic with $\pi_q(R_{p-1}) \times \pi_r(R_{p-1})$. In general, however, $\pi_{q,r}(R_{p-1})$ is not abelian.

The group R_1 is naturally homeomorphic with S^1 ; the group multiplication in R_1 defines a map $f: S^1 \times S^1 \rightarrow R_1$ representing an element $\alpha \in \pi_{1,1}(R_1)$. It turns out that $J'(\alpha)$ is the nonzero element of $\pi_4(S^2)$.

We may consider R_{p-1} as a subgroup of R_p ; the inclusion map defines a homomorphism $\beta: \pi_{q,r}(R_{p-1}) \rightarrow \pi_{q,r}(R_p)$. Then the commutativity relation $J' \circ \beta = E \circ J'$, suggested by the diagram

$$\begin{array}{ccc} \pi_{q,r}(R_{p-1}) & \xrightarrow{\beta} & \pi_{q,r}(R_p) \\ \downarrow J' & & \downarrow J' \\ \pi_{p+q+r}(S^p) & \xrightarrow{E} & \pi_{p+q+r+1}(S^{p+1}) \end{array}$$

holds. Hence the nonzero element of $\pi_{p+2}(S^p)$ can be obtained from $\pi_{1,1}(R_{p-1})$ by means of the operation J' .

The operation J' has the following properties:

- (1) $J' | \pi_{q+r}(R_{p-1}) = J$;
- (2) $J'(\pi_q(R_{p-1})) = J'(\pi_r(R_{p-1})) = 0$;
- (3) if $\alpha \in \pi_q(R_{p-1})$, $\beta \in \pi_r(R_{p-1})$, then the element $J'(\alpha \circ \beta)$ depends bilinearly on α and β .

In particular, J' is not in general a homomorphism.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
CAMBRIDGE, MASS., U. S. A.

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FIBRE BUNDLES AND OBSTRUCTIONS

THE THEORY OF OBSTRUCTIONS

PAUL OLUM

1. Introduction. In the study of the mappings of one topological space into another, there are a number of rather different problems which seem to be most effectively studied by a certain common technique. Examples are the extension problem, the homotopy problem, the deformation problem (that is, to deform a single given mapping so that after the deformation it behaves in some prescribed fashion), and the cross-section or "lifting" problem in fibre bundles. The common technique in question is the method of "obstructions." An obstruction is simply the algebraic expression, in terms of cohomology theory, of the difficulty one runs into in trying to perform the required extension or homotopy or deformation or lifting. The first explicit formulation of the notion of an obstruction is due to Eilenberg [1], although the method had been used implicitly earlier by Whitney [6] in his restatement and proof of the Hopf mapping theorem in cohomology terms.

In view of the limitations of time, we shall confine ourselves here to a rather specific problem: Given two arcwise connected spaces X , Y with chosen base points x_0 , y_0 , we consider mappings $X \rightarrow Y$ which carry x_0 into y_0 and study their homotopy relative to x_0 . What I shall have to say about obstructions for this problem will be fairly representative also of the situation for free-homotopy or homotopy relative to an arbitrary connected subspace, and for the other problems mentioned earlier, although there are, of course, certain important differences.

We consider then two mappings $f_0, f_1: X \rightarrow Y$ with $f_0(x_0) = f_1(x_0) = y_0$. I propose first of all to define the n th obstruction to a homotopy of f_0 to f_1 relative to x_0 , which we shall denote $\mathcal{O}^n(f_0, f_1) \text{ rel. } x_0$, then to indicate a few of the most important properties of these obstructions; finally, in the remaining time, I shall describe some new results in an essentially classical vein—having to do with mappings of manifolds and the notion of degree—which are proved using the obstruction technique.

2. Cohomology over a local group. We begin by supposing there is given a local system of groups $\mathcal{G} = \{G_\nu\}$ in the space Y , as used by Steenrod in his paper on homology with local coefficients [4]. A mapping $f: X \rightarrow Y$ induces a local system $\mathcal{G}(f)$ in X , by assigning to each $x \in X$ the group $G_{f(x)}$ and to each path in X the isomorphism attached to the image path in Y . With this, we may then consider the cohomology groups $H^n(Y, \mathcal{G})$ and $H^n(X, \mathcal{G}(f))$.

We introduce one further definition here. By a *local group* G at the point y_0 in Y we shall mean a group G together with an operation of the fundamental group $\pi_1(Y, y_0)$ on it. Clearly, in the local system $\mathcal{G} = \{G_\nu\}$, the group at the

base point G_{y_0} may be regarded as a local group at y_0 . Now it is easily seen that the cohomology group $H^n(Y, \mathcal{G})$ does not really depend on the full spread of the local system $\mathcal{G}$ but is completely determined to within a unique isomorphism by the local group G_{y_0} carried by $\mathcal{G}$. This leads (we are omitting the obvious formal details) to the definition of cohomology $H^n(Y, G_{y_0})$ over the local group G_{y_0} , and this cohomology group is isomorphic in a unique way to $H^n(Y, \mathcal{G})$.

Now it is to be observed that the local group at x_0 in X carried by $\mathcal{G}(f)$ does not depend on all of $\mathcal{G}$ and on f itself, but only on the local group G_{y_0} and on the homomorphism $\theta: \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ determined by f . Let us then denote this local group at x_0 in X , induced by θ , by $\theta^*G_{y_0}$. Thus we are led to consider the group $H^n(X, \theta^*G_{y_0})$ which is related to $H^n(X, \mathcal{G}(f))$ by a unique isomorphism.

As is well-known, the homotopy groups $\pi_n(Y, y)$ form a local system of groups in Y and this is the particular case of the local system $\mathcal{G}$ which concerns us. Given any $\theta: \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$, we have induced in X the local group $\theta^*\pi_n(Y, y_0)$, which we shall shorten simply to $\theta^*\pi_n$, and we may then consider the cohomology group $H^n(X, \theta^*\pi_n)$. It is this cohomology group which appears to be the appropriate algebraic vehicle for carrying the obstruction technique.

3. Definition of the obstruction. We now proceed to define the obstruction $\Theta^n(f_0, f_1)$ rel. x_0 which, as we shall see, will turn out to be a subset (possibly a void subset) of the group $H^n(X, \theta_0^*\pi_n)$, where θ_0 is the homomorphism induced by f_0 . (Details concerning the material of this and the following section will be found in [3].)

This obstruction can be defined completely invariantly and under the most general circumstances, with essentially no limitations on the spaces X and Y under consideration other than arcwise connectedness; see [3]. Here again, however, in the interests of simplicity and time, I shall suppose that X is triangulable and has been decomposed to form a simplicial complex K ; it will be assumed also that the vertices of K are partially ordered so that the vertices of any simplex are simply ordered.

We consider then the cartesian product space $I \times X = \bar{X}$, where I is the unit interval, and suppose it also triangulated so that the triangulation (including the partial ordering of vertices) agrees with that already given on $0 \times X$ and $1 \times X$. We identify X with $0 \times X$, denote by $\bar{K}$ the complex of $\bar{X} = I \times X$ and by K_1 the complex of $1 \times X$. We define a mapping $\bar{f}_{01}: X \cup (I \times x_0) \cup (1 \times X) \rightarrow Y$ as follows: We map X by f_0 , $1 \times X$ by f_1 , and we map $I \times x_0$ all into y_0 .

Now suppose $\bar{f}$ is an extension of this partial mapping $\bar{f}_{01}$ over $\bar{X}^n$, the space of the n -skeleton of $\bar{X}$, where $n \geq 2$. (Of course, there may be no such $\bar{f}$). Let us consider any $(n+1)$ -simplex $\bar{\sigma}_{n+1}$ of $\bar{X}$ with leading vertex $\bar{x}$. The mapping $\bar{f}$ confined to the boundary of $\bar{\sigma}_{n+1}$ then defines an element of $\pi_n(Y, \bar{f}(\bar{x}))$, and, in particular, for $\bar{\sigma}_{n+1} \in K$ or K_1 , the element so defined is clearly the zero element of the group. Consequently, if $\pi_n(\bar{f})$ denotes the local system of groups in $\bar{X}$ induced by $\bar{f}$ from the local system $\pi_n(Y, y)$, then this means that $\bar{f}$ defines an

element, a cochain, of $C^{n+1}(\bar{K}, K \cup K_1, \pi_n(\tilde{f}))$. (Actually, the local system $\pi_n(\tilde{f})$ is not defined over all of $\bar{X}$, only on the subspace $\bar{X}^n$, but this is sufficient since for cohomology in $\bar{K}$ the local system need be given only on the 2-skeleton.)

We come now to the essential observation that this cochain is indeed a cocycle. This is, of course, analogous to the corresponding result in Eilenberg's original work; see [1]. It is basically a consequence of the definition of addition in the homotopy groups and of the isomorphisms of homotopy groups associated with paths in Y . In this way then $\tilde{f}$ defines a cocycle and consequently an element $\bar{h}^{n+1}(\tilde{f})$ of $H^{n+1}(\bar{K}, K \cup K_1, \pi_n(\tilde{f}))$.

Next, with the object of "pulling" this cohomology element down into the base space X , we look at the homomorphism:

$$\Delta: H^n(K, \pi_n(f_0)) \rightarrow H^{n+1}(\bar{K}, K \cup K_1, \pi_n(\tilde{f}))$$

which is defined by the coboundary operator; we have used here also the fact that $\pi_n(\tilde{f})$ confined to X is just the local system $\pi_n(f_0)$ induced by the mapping f_0 . This Δ is easily proved to be an isomorphism onto by imbedding it in an appropriate exact sequence in which the third term is clearly zero. This being so, we may apply Δ^{-1} to the element $\bar{h}^{n+1}(\tilde{f})$ and get an element $\Delta^{-1}(\bar{h}^{n+1}(\tilde{f})) \in H^n(K, \pi_n(f_0))$. As we observed at the outset, $H^n(K, \pi_n(f_0)) \approx H^n(K, \theta_0^* \pi_n)$ where $f_0: X \rightarrow Y$ induces θ_0 . Thus $\Delta^{-1}(\bar{h}^{n+1}(\tilde{f}))$ determines a unique element in $H^n(K, \theta_0^* \pi_n)$ which we denote by $h^n(\tilde{f})$.

Now consider the totality $\{h^n(\tilde{f})\}$ of *distinct* elements so defined for all possible extensions $\tilde{f}$ over $\bar{X}^n$. These elements form a subset (possibly void) of $H^n(K, \theta_0^* \pi_n)$ which we denote $\Theta^n(f_0, f_1)$ rel. x_0 and call *the n th obstruction to a homotopy of f_0 to f_1 relative to x_0* . Note that $n \geq 2$ here; the role of the obstruction for the lowest significant dimension, $n = 1$, is played by the homomorphisms of fundamental groups induced by the mappings, as is made clear by Theorem I below.

4. Properties of obstructions. The principal geometric significance of these obstructions is contained in the following theorem which makes clear their precise relation to the homotopy problem and which justifies calling them "obstructions":

THEOREM I. *For fixed n the following statements are equivalent. Each statement applies only for the indicated range of n :*

(a) ($n = 1$) $\theta_0 = \theta_1$; ($n \geq 2$) $\Theta^n(f_0, f_1)$ rel. x_0 is nonvoid and contains the zero element.

(b) ($n \geq 1$) $f_0 \cong f_1$ rel. x_0 (dim. n).¹

(c) ($n \geq 1$) $\Theta^{n+1}(f_0, f_1)$ rel. x_0 is nonvoid.

The principal algebraic property of obstructions is given by the following theorem which I shall call the "addition theorem" for obstructions:

¹ This means " $f_0|X^n$ is homotopic to $f_1|X^n$ relative to x_0 " where X^n is the space of the n -skeleton of K .

THEOREM II. Given f_0, f_1, f_2 , suppose $\Theta^n(f_0, f_1)$ rel. x_0 and $\Theta^n(f_1, f_2)$ rel. x_0 are nonvoid (so that $\theta_0 = \theta_1 = \theta_2$) and contain, respectively, h_0^n and h_1^n . Then

$$\begin{aligned}\Theta^n(f_0, f_2) \text{ rel. } x_0 &= h_0^n + \Theta^n(f_1, f_2) \text{ rel. } x_0 \\ &= \Theta^n(f_0, f_1) \text{ rel. } x_0 + h_1^n \\ &= \Theta^n(f_0, f_1) \text{ rel. } x_0 + \Theta^n(f_1, f_2) \text{ rel. } x_0.\end{aligned}$$

With one exception, to which we shall return shortly, essentially everything of a general character about these obstructions and their relevance for the homotopy problem is immediately contained in Theorems I and II. Note here, in Theorem II, the importance of our using cohomology over the local group $\theta^*\pi_n$ rather than over the local system $\pi_n(f)$ induced by f ; this is necessary in order for the addition in Theorem II to be defined.

We mention a few particular consequences contained in Theorems I and II:

- (1) $\Theta^n(f_0, f_0)$ rel. x_0 is a subgroup of $H^n(K, \theta_0^*\pi_n)$.
- (2) $\Theta^n(f_0, f_1)$ rel. x_0 is (if nonvoid) a coset of this subgroup. (Thus $\Theta^n(f_0, f_1)$ rel. x_0 and $\Theta^n(f_0, f_2)$ rel. x_0 are either disjoint or identical.)
- (3) Suppose f_0, f_1, f_2 homotopic in dimension $n - 1$ rel. x_0 . Then $f_1 \cong f_2$ rel. x_0 (dim. n) if and only if

$$\Theta^n(f_0, f_1) \text{ rel. } x_0 = \Theta^n(f_0, f_2) \text{ rel. } x_0.$$

This last statement has the importance that, out of a set of mappings homotopic in dimension $n - 1$, homotopy in dimension n can be studied by comparing each of the mappings with a fixed one, f_0 .

Returning now to the exception referred to just above, the other piece of information we need about obstructions is an existence theorem:

THEOREM III. Given $f_0: X \rightarrow Y$ inducing θ_0 and $h_0^n \in H^n(K, \theta_0^*\pi_n)$, there exists $f_1: X^{n+1} \rightarrow Y$ such that $\Theta^n(f_0, f_1)$ rel. x_0 contains h_0^n .

The proof of this theorem is given by taking a cocycle $z_0^n \in Z^n(K, \pi_n(f_0))$ representing h_0^n and then modifying f_0 , keeping it fixed on X^{n-1} , but "altering" it inside of each n -cell σ_n , in a certain well-defined fashion, by the homotopy element $z_0^n(\sigma_n)$. The resulting mapping f_1 is extendable over X^{n+1} and is easily proved to have the property stated in the theorem.

The corresponding existence theorem in the lowest dimension says that for any θ there is a mapping of the 2-skeleton inducing it.

Property (3) above and this existence theorem combine to yield what may be called the general "one-stage" homotopy enumeration theorem, namely: If one considers a class of mappings of X^{n+1} into Y , all homotopic in dimension $n - 1$ rel. x_0 , and singles one of them, f_0 , out for reference, then the homotopy classes in dimension n are in 1-1 correspondence with the elements of the factor group $H^n(K, \theta_0^*\pi_n)/\Theta^n(f_0, f_0)$ rel. x_0 . This indicates the central role played by the "self-obstruction" $\Theta^n(f_0, f_0)$ rel. x_0 in the study of homotopy classifications. As an example, Steenrod's classification of the maps of an $(n + 1)$ -complex into

the n -sphere [5] amounts in essence to the determination, for any such mapping, of this self-obstruction, which he showed can be expressed as a certain cup-sub- i product.

There is one particular case in which, by the theorems above (and a simple extension theorem corresponding to I), we have the complete homotopy enumeration result already at hand. This is the case where $X = X^n$ is an n -polyhedron and $\pi_r(Y, y_0) = 0$ for $1 < r < n$. Then

THEOREM IV. *The homotopy classes rel. x_0 of mappings $X \rightarrow Y$ are in 1-1 correspondence with the pairs (θ, h) where θ is any homomorphism of the fundamental groups and $h \in H^n(X, \theta^*\pi_n)$.*

The corresponding result for free-homotopy is a little more complicated. It can be written down equally explicitly, but involves an additional invariant, the Eilenberg-MacLane invariant k^{n+1} for the space Y .

5. Remarks. Theorem IV, as it stands, is simply an enumeration theorem; it tells "how many" homotopy classes there are. However, with an appropriate re-interpretation it can be much more than that. The key to this is provided by the existence Theorem III described earlier. In the proof of this theorem, which was indicated very briefly above, one has not only the existence of a mapping f_1 so that a prescribed coset is realized as an obstruction, but also a rather specific description of f_1 as an "alteration" of f_0 by certain homotopy elements. This, combined with Theorem IV, then gives what we might call a "homotopy representation theorem"; i.e., an enumeration of homotopy classes together with the presentation of a particular representative mapping from each homotopy class.

One can then hope to gain a deeper insight into the properties which characterize the mappings of a given homotopy class by a direct examination of these representative mappings. In this sense I think a theorem like Theorem IV may be regarded not only as a result but also, and even primarily, as a starting point for studying homotopy properties of mappings.

6. Mappings of manifolds and the twisted degree. I would like to leave the general discussion of obstructions now and try to illustrate these last remarks by describing in the remainder of this paper some results connected with mappings of manifolds and the notion of degree. I shall not refer to obstructions again except to say now that the results to be described are in large part consequences of this homotopy representation theorem. Proofs and further details are contained in a paper which will appear later.

We consider two connected n -manifolds M^n and Q^n (assumed triangulable) and we study the mappings $M^n \rightarrow Q^n$ for the case where $\pi_r(Q^n, y_0) = 0$ for $1 < r < n$. All mappings carry $x_0 \in M^n$ into $y_0 \in Q^n$.

Before stating the relevant theorems, I want to extend somewhat Brouwer's

original definition of degree to what I shall call the *twisted* degree. This will be done here in invariant fashion; there is a corresponding extremely simple geometric interpretation which we shall not take time for.

For this definition, we take M^n and Q^n to be compact and make no restriction on the homotopy groups of Q^n . Let I_0^t denote the *local group of twisted integers* at y_0 in Q^n ; this means the integers I_0 together with the following operation of the fundamental group $\pi_1(Q^n, y_0)$ on them: an orientation-preserving element operates trivially, an orientation-reversing element operates nontrivially.

Let $f: M^n \rightarrow Q^n$ induce θ . Then we may consider the cohomology groups $H^n(Q^n, I_0^t)$ and $H^n(M^n, \theta^*I_0^t)$, and f induces, in obvious fashion, also a homomorphism:

$$(6.1) \quad m_f: H^n(Q^n, I_0^t) \rightarrow H^n(M^n, \theta^*I_0^t).$$

Now $H^n(Q^n, I_0^t)$ is easily seen to be infinite cyclic. Let us suppose for the moment that the homomorphism θ is *orientation-true*, that is, it carries orientation-preserving elements into orientation-preserving elements, orientation-reversing into orientation-reversing. Then $\theta^*I_0^t$ is clearly just the local group of twisted integers at x_0 in M^n and the group on the right is also infinite cyclic. Assuming that one of the two possible generators has been selected once and for all for these n th cohomology groups over the twisted integers, then the homomorphism m_f is completely described by an integer which we call the *twisted degree* of f and denote $\deg f$.

This twisted degree has the usual properties, transitivity, etc. One remark, however: If $f_0 \cong f_1$, then $\deg f_0 = \deg f_1$ or $-\deg f_1$ according as the image of x_0 , during the homotopy, traces out an orientation-preserving or orientation-reversing path in Q^n .

If the homomorphism θ is not orientation-true, then the group on the right in (6.1) becomes cyclic order 2 and the twisted degree is an integer mod 2. In this case it is identical with the ordinary mod 2 degree.

7. Homotopy properties of mappings of manifolds. We now return to our original problem of the mappings $M^n \rightarrow Q^n$ with $\pi_r(Q^n, y_0) = 0$ for $1 < r < n$.

If either M^n or $\tilde{Q}^n$ (the universal covering space of Q^n) is noncompact, then the classification of these mappings is particularly simple: Two mappings are homotopic rel. x_0 [or, free-homotopic] if and only if they induce the same homomorphism [respectively, homomorphism-class]² of the fundamental group.

We may as well suppose then M^n and $\tilde{Q}^n$ compact. This means, in particular, that the fundamental group $\pi_1(Q^n, y_0)$ has finite order; let us call it l . Now let $f_0, f_1: M^n \rightarrow Q^n$ induce θ_0, θ_1 . There are two cases here:

Case 1. Orientation-true θ 's. For this case we have the following theorem:

THEOREM V. $f_0 \cong f_1$ rel. x_0 if and only if $\theta_0 = \theta_1$ and $\deg f_0 = \deg f_1$. Further-

² Two homomorphisms belong to the same homomorphism-class if they differ by an inner automorphism of the image group.

more, for each θ , the associated twisted degrees are all of the integers of a single congruence class mod l .

The corresponding result for free-homotopy is an immediate consequence of Theorem V and the remark made earlier on the behavior of the twisted degree under a homotopy. It is given, in detailed fashion, by

THEOREM Va. $f_0 \cong f_1$ with the image of x_0 moving through a path of $\omega \in \pi_1(Q^n, y_0)$ if and only if $\theta_0(\gamma) = \omega \theta_1(\gamma) \omega^{-1}$ for all $\gamma \in \pi_1(M^n, x_0)$ and

(a) (for ω orientation-preserving): $\deg f_0 = \deg f_1$.

(b) (for ω orientation-reversing): $\deg f_0 = -\deg f_1$.

Case 2. Non-orientation-true θ 's. If the homomorphisms are not orientation-true, then we no longer have so satisfactory a characterization for the homotopy class of a mapping. However, this is largely compensated for by a very great reduction in the number of distinct homotopy classes:

THEOREM VI. For each θ , there are exactly two distinct homotopy classes rel. x_0 inducing θ .

In general, for this non-orientation-true case, the twisted degree (or, what is the same thing, the degree mod 2) will not distinguish between the two classes. However, if l is odd (which means Q^n is orientable), then it will do so for this case also.

In the corresponding case of free-homotopy there are either one or two homotopy classes for each homomorphism-class² Θ of the fundamental groups. The result here is, however, not an immediate consequence of Theorem VI, unless l is odd, and a particular computation is required to settle the matter in any given circumstance. This computation is quite explicit and straightforward, but we shall not consider it here.

8. Homotopy-type. An important application of Theorem V is to the problem of homotopy-type for manifolds M^n, Q^n satisfying $\pi_r(M^n, x_0) = \pi_r(Q^n, y_0) = 0$ for $1 < r < n$. It is easily seen that a homotopy equivalence between two manifolds must induce an *orientation-true* isomorphism of the fundamental groups so that Theorem V is the relevant one.

A particular case worth noting in this connection is that of the higher dimensional lens spaces. Given two $(2n + 1)$ -dimensional lens spaces $L = L(m; q_1, \dots, q_n)$ and $L' = L(m'; q'_1, \dots, q'_n)$, then, for any homomorphism θ of their fundamental groups, it is quite easy to write down one particular mapping of L into L' inducing θ and to determine its degree. An immediate consequence then is the homotopy-type classification of the higher dimensional lens spaces:

THEOREM VII. L and L' have the same homotopy-type if and only if $m = m'$ and

$$q_1 q_2 \cdots q_n \equiv \pm k^2 q'_1 \cdots q'_n \pmod{m}$$

for some k relatively prime to m .

9. The group-ring degree. I want to make one final remark here concerning the notion of the degree of a mapping. It is possible to generalize the degree still further, and I think quite fruitfully, by using in the definition given above in §6, instead of the local group I_0^t , the group-ring of the fundamental group of Q^n over the integers.

If one does so, he finds that the homomorphism (6.1) can again be characterized by an integer (or an integer mod 2 for the special case that θ carries some orientation-reversing element into the unit element) even though the group on the right in (6.1) will no longer be infinite cyclic. This integer might then be called the "group-ring degree" of the mapping. It agrees with the twisted degree if θ is orientation-true and also includes, in the general case, the Hopf notion of the "Absolutgrad" of a mapping [2]; in fact, the "Absolutgrad" is just the absolute value of the group-ring degree.

This group-ring degree will help to distinguish between the two homotopy classes (Theorem VI) in some, but by no means all, of the cases where the twisted degree fails to do so.

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CORNELL UNIVERSITY,
ITHACA, N. Y., U. S. A.

HOMOTOPY GROUPS OF TRIADS

W. S. MASSEY

1. Introduction. One of the most important problems of topology today is to devise methods for computing the homotopy groups of topological spaces. The present paper is an attack on one phase of this general problem.

One method of attacking the problem of determining the homotopy groups of a connected cell complex K is first to try to determine the relative homotopy groups $\pi_p(K^q, K^{q-1})$, where K^n denotes the n -skeleton of K . A slightly more general problem is to try to determine the relative homotopy groups $\pi_p(X^*, X)$, where X is a pathwise connected topological space, and X^* is obtained by the adjunction of q -cells, $\varepsilon_1^q, \varepsilon_2^q, \dots$, to X ; i.e., $X^* = X \cup \varepsilon_1^q \cup \varepsilon_2^q \cup \dots$. This general problem will be the main concern of this paper.

The analogous problem from homology theory, namely, to compute the relative homology groups, $H_p(K^q, K^{q-1})$, or $H_p(X^*, X)$, can be solved completely, and when applied to the problem of determining the homology groups of the complex K , leads to the usual algorithm for computing homology groups by means of chains, cycles, bounding cycles, etc. This comparison naturally leads one to ask why the homology groups $H_p(X^*, X)$ can be determined, while the problem of determining the homotopy groups $\pi_p(X^*, X)$ remains unsolved. The answer to this question is that the so-called "excision property" (see Eilenberg and Steenrod [2]) holds for homology groups, but does not hold for homotopy groups.

The excision property may be formulated as follows. Let $X = A \cup B$; then under rather general conditions the homomorphism $i_*: H_p(A, A \cap B) \rightarrow H_p(X, B)$, induced by the inclusion map $i: (A, A \cap B) \rightarrow (X, B)$, is an isomorphism onto in all dimensions. Simple examples may be given to show that this property does not hold when homology groups are replaced by homotopy groups, even if X is a finite, connected simplicial complex, and A, B , and $A \cap B$ are connected subcomplexes. Although the excision property does not hold in general for homotopy groups, under certain special conditions the homomorphisms $i_*: \pi_p(A, A \cap B) \rightarrow \pi_p(X, B)$ are isomorphisms onto for special values of p , and even when these homomorphisms fail to be isomorphisms onto, it is advantageous to determine as much information as possible about them. The theory developed in this paper enables one to fit the homomorphisms

$$i_*: \pi_p(A, A \cap B) \rightarrow \pi_p(X, B),$$

for $p = 2, 3, 4, \dots$, into an exact sequence of homomorphisms in which the remaining groups are the new homotopy groups of the "triad", $(X; A, B)$, denoted by $\pi_p(X; A, B)$. The exact sequence in question is as follows:

$$\dots \rightarrow \pi_{p+1}(X; A, B) \rightarrow \pi_p(A, A \cap B) \xrightarrow{i_*} \pi_p(X, B) \rightarrow \pi_p(X; A, B) \rightarrow \dots$$

This enables one to translate theorems about the homomorphisms

$$i_*: \pi_p(A, A \cap B) \rightarrow \pi_p(X, B)$$

into theorems about the triad homotopy groups and vice-versa.

One important case where a homomorphism induced by an excision has been studied is the following. Let S^n be an n -sphere, E_+^n and E_-^n the upper and lower "hemispheres," and S^{n-1} the "equatorial" $(n-1)$ -sphere, $S^{n-1} = E_+^n \cap E_-^n$. Then the homomorphism $i_*: \pi_p(E_+^n, S^{n-1}) \rightarrow \pi_p(S^n, E_-^n)$ is equivalent (in a way to be explained later) to the "Einhängung", or suspension, $E: \pi_{p-1}(S^{n-1}) \rightarrow \pi_p(S^n)$, first studied by Freudenthal [3]. It was the study of this special case which first suggested to the author the idea for the homotopy groups of a triad.

The author wishes to acknowledge that all the results in this paper were obtained in collaboration with A. L. Blakers; this is actually a report on joint work.

2. Definition of the homotopy groups of a triad. We shall use the following notation for certain subsets of Cartesian n -space, C^n . The coordinates of a point $x \in C^n$ are denoted by $(x_1, \dots, x_n)$, and $|x| = (x_1^2 + \dots + x_n^2)^{1/2}$.

$$\begin{aligned} E^n &= \{x \in C^n \mid |x| \leq 1\}, & S^{n-1} &= \{x \in E^n \mid |x| = 1\}, \\ E_+^{n-1} &= \{x \in S^{n-1} \mid x_n \geq 0\}, & E_-^{n-1} &= \{x \in S^{n-1} \mid x_n \leq 0\}, \\ S^{n-2} &= \{x \in S^{n-1} \mid x_n = 0\}, & E_1^n &= \{x \in E^n \mid x_2 \geq 0\}, \\ E_2^n &= \{x \in E^n \mid x_2 \leq 0\}, & p_0 &= (1, 0, \dots, 0). \end{aligned}$$

Let $(X; A, B)$ be a *triad*; that is, A and B are subspaces of the topological space X , and the intersection, $A \cap B$, is nonvoid. Choose a base point $x_0 \in A \cap B$. A map

$$f: (E^n; E_+^{n-1}, E_-^{n-1}, p_0) \rightarrow (X; A, B, x_0)$$

is a continuous function $f: E^n \rightarrow X$ such that $f(E_+^n) \subset A$, $f(E_-^n) \subset B$, and $f(p_0) = x_0$. Two such maps f_0, f_1 are *homotopic* if they are connected by a continuous 1-parameter family of maps,

$$f_t: (E^n; E_+^{n-1}, E_-^{n-1}, p_0) \rightarrow (X; A, B, x_0),$$

where $0 \leq t \leq 1$. With these definitions, it is not difficult to prove the following:

LEMMA 1. *If $n \geq 3$, and $f: (E^n; E_+^{n-1}, E_-^{n-1}, p_0) \rightarrow (X; A, B, x_0)$ is a given map, then there exist homotopic maps f', f'' , such that $f'(E_1^n) = f''(E_2^n) = x_0$.*

Denote the set of all homotopy classes of maps $(E^n; E_+^{n-1}, E_-^{n-1}, p_0) \rightarrow (X; A, B, x_0)$ by $\pi_n(X; A, B, x_0)$. For $n \geq 3$, we define an addition between any two elements $\alpha, \beta \in \pi_n(X; A, B, x_0)$ as follows: choose maps f, g belonging to the homotopy classes α, β , respectively, with $f(E_2^n) = g(E_1^n) = x_0$. The existence of representatives f and g satisfying these conditions is guaranteed by Lemma 1.

Define $h: (E^n; E_+^{n-1}, E_-^{n-1}, p_0) \rightarrow (X; A, B, x_0)$ by $h|E_1^n = f|E_1^n$ and $h|E_2^n = g|E_2^n$. Then $\alpha + \beta$ is defined to be the homotopy class of h . With this definition of addition, $\pi_n(X; A, B, x_0)$ becomes a group, called the n th homotopy group of the triad $(X; A, B)$ at the base point x_0 .

The following are some of the simpler properties of the groups thus defined:

(a) $\pi_n(X; A, B, x_0)$ is abelian for $n > 3$; simple examples show that it need not be so for $n = 3$.

(b) The system of groups $\pi_n(X; A, B, x)$ for $x \in A \cap B$ forms a local system of groups in the space $A \cap B$ in the sense of Steenrod, and $\pi_1(A \cap B, x_0)$ is a group of operators on $\pi_n(X; A, B, x_0)$.

(c) If $A \supset B$, then $\pi_n(X; A, B)$ is isomorphic to $\pi_n(X, A)$.

(d) A continuous map $f: (X; A, B, x_0) \rightarrow (X'; A', B', x'_0)$ induces homomorphisms of the corresponding triad homotopy groups. We denote these by $f_*: \pi_p(X, A, B, x_0) \rightarrow \pi_p(X'; A', B', x'_0)$.

3. The exact sequences of a triad. Associated with a triad $(X; A, B)$ are two boundary homomorphisms,

$$\partial_+: \pi_n(X; A, B, x_0) \rightarrow \pi_{n-1}(A, A \cap B, x_0),$$

$$\partial_-: \pi_n(X; A, B, x_0) \rightarrow \pi_{n-1}(B, A \cap B, x_0),$$

defined as follows. If $f: (E^n; E_+^{n-1}, E_-^{n-1}, p_0) \rightarrow (X; A, B, x_0)$ represents

$$\alpha \in \pi_n(X; A, B, x_0) \quad (n \geq 3),$$

then the maps

$$f_+: (E_+^{n-1}, S^{n-2}, p_0) \rightarrow (A, A \cap B, x_0),$$

$$f_-: (E_-^{n-1}, S^{n-2}, p_0) \rightarrow (B, A \cap B, x_0),$$

defined by f are representatives of the elements $\partial_+(\alpha) \in \pi_{n-1}(A, A \cap B, x_0)$ and $\partial_-(\alpha) \in \pi_{n-1}(B, A \cap B, x_0)$ respectively (assuming, of course, that a suitable convention is adopted to determine the orientations of the cells E_+^{n-1} and E_-^{n-1}).

Now consider the following sequence of groups and homomorphisms:

$$\begin{aligned} \cdots \xrightarrow{\partial_+} \pi_n(A, A \cap B) \xrightarrow{i_*} \pi_n(X, B) \xrightarrow{j_*} \\ \pi_n(X; A, B) \xrightarrow{\partial_+} \pi_{n-1}(A, A \cap B) \longrightarrow \cdots \end{aligned}$$

The homomorphisms i_* and j_* are induced by the inclusion maps

$$i: (A, A \cap B) \rightarrow (X, B), \quad j: (X; x_0, B) \rightarrow (X; A, B).$$

This sequence is one of the two homotopy sequences of the triad $(X; A, B)$; the other sequence is obtained by interchanging the roles of A and B throughout.

THEOREM 1. *The homotopy sequences of a triad are both exact.*

The proof of this theorem requires only a straightforward verification of exactness, using the definitions already given.

In case $X = A \cup B$, the inclusion map $i: (A, A \cap B) \rightarrow (X, B)$ is an excision, and the triad homotopy groups may be considered to be a measure of the amount by which the relative homotopy groups fail to satisfy the excision axiom, as was indicated in the introduction.

4. The homotopy groups of the triad $(X^*; \mathcal{E}^n, X)$. For the remainder of this paper we shall assume that X^* is a Hausdorff space and X is a closed, pathwise connected subset of X^* such that there exists a decomposition

$$X^* - X = U_1^n \cup U_2^n \cup \dots \cup U_k^n$$

of $X^* - X$ into k disjoint open sets U_i^n having the following property. Denote the closure of U_i^n by $\mathcal{E}_i^n$, and let $\mathcal{E}^n = X \cap \mathcal{E}^n$, then it is assumed that there exist continuous mappings

$$\psi_i: (E^n, S^{n-1}) \rightarrow (\mathcal{E}_i^n, \mathcal{E}_i^n), \quad i = 1, \dots, k,$$

which map $E^n - S^{n-1}$ homeomorphically onto U_i^n . It follows that S^{n-1} is mapped onto $\mathcal{E}_i^n$ by ψ_i , but not necessarily homeomorphically. Under these conditions we shall say that X^* is obtained from X by the adjunction of the cells $\mathcal{E}_1^n, \dots, \mathcal{E}_k^n$. This is exactly the process by means of which a general cell complex is built. The following notation will be used consistently:

$$\mathcal{E}^n = \mathcal{E}_1^n \cup \mathcal{E}_2^n \cup \dots \cup \mathcal{E}_k^n, \quad \mathcal{E}^n = \mathcal{E}_1^n \cup \dots \cup \mathcal{E}_k^n = X \cap \mathcal{E}^n.$$

We shall always make the additional assumption that $\mathcal{E}^n$ is arcwise connected. This last condition is not nearly as restrictive as it appears on first sight, since J. H. C. Whitehead has shown that the homotopy type of the space X^* depends only on the homotopy class of the maps $\psi'_i: S^{n-1} \rightarrow X$ defined by the maps $\psi_i: (E^n, S^{n-1}) \rightarrow (\mathcal{E}_i^n, \mathcal{E}_i^n)$, $i = 1, 2, \dots, k$ (see [7] and [8]).

As stated in the introduction, our main problem is to determine the relative homotopy groups $\pi_p(X^*, X)$. This problem leads naturally to consideration of the excision homomorphism $\pi_p(\mathcal{E}^n, \mathcal{E}^n) \rightarrow \pi_p(X^*, X)$, and hence to consideration of the homotopy sequence of the triad $(X^*; \mathcal{E}^n, X)$. It is the purpose of this section to state two theorems of a comparatively elementary nature which often give considerable information about this sequence, and hence about the groups $\pi_p(X^*, X)$.

First, we need some definitions. A topological space Y is said to be *n-connected* ($n > 0$) if it is pathwise connected and $\pi_p(Y) = 0$ for $1 \leq p \leq n$. A pair (Y, A) is said to be *n-connected* ($n \geq 1$) if both Y and A are pathwise connected, the injection $\pi_1(A) \rightarrow \pi_1(Y)$ is a homomorphism onto, and $\pi_p(Y, A) = 0$ for $1 < p \leq n$. Similarly, we shall say a triad $(Y; A, B)$ is *n-connected* ($n \geq 2$) if both of the pairs $(A, A \cap B)$ and $(B, A \cap B)$ are 1-connected, the injections $\pi_2(A, A \cap B) \rightarrow \pi_2(Y, B)$ and $\pi_2(B, A \cap B) \rightarrow \pi_2(Y, A)$ are homomorphisms onto, and $\pi_p(Y; A, B) = 0$ for $2 < p \leq n$.

THEOREM 2. *If the pair $(X, \mathcal{E}^n)$ is m -connected, $m \geq 1$ and $n > 2$, then the triad $(X^*; \mathcal{E}^n, X)$ is $(m + n - 1)$ -connected. (In case $m = 1$, it is also necessary to assume that $\pi_2(X, \mathcal{E}^n)$ is abelian.)*

THEOREM 3. *If the space $\mathcal{E}^n$ is m -connected, $m \geq 1$, then the boundary homomorphism*

$$\partial: \pi_{p+1}(X^*; \mathcal{E}^n, X) \rightarrow \pi_p(\mathcal{E}^n, \mathcal{E}^n)$$

is the zero homomorphism for $2 \leq p \leq m + n - 1$.

The proof of these theorems is too long to be given here; for full details, see [1]. Theorem 2 is closely related to what J. H. C. Whitehead calls the “crude suspension theorem” (see [9]) and can be proved by his methods. These methods are highly geometrical, and go back to Freudenthal [3]. The author’s proofs make more use of the techniques of algebraic topology, and are not so geometrical in character. In particular, use is made of the basic results of the theory of “obstructions” to extensions and deformations of continuous maps.

As an example of the application of these results, consider the triad $(S^n; E_+^n, E_-^n)$. By applying Theorem 2, we see that this triad is $(2n - 2)$ -connected. It follows from the exactness of the homotopy sequence of this triad that the excision homomorphism $i_*: \pi_p(E_+^n, S^{n-1}) \rightarrow \pi_p(S^n, E_-^n)$ is an isomorphism onto if $p < 2n - 2$, and is a homomorphism onto if $p = 2n - 2$. In this case, the excision homomorphism is equivalent to the suspension or “Einhängung” homomorphism introduced by Freudenthal [3]; in fact, in the following diagram

$$\begin{array}{ccc} \pi_{p-1}(S^{n-1}) & \xrightarrow{E} & \pi_p(S^n) \\ \partial \uparrow & & \downarrow k_* \\ \pi_p(E_+^n, S^{n-1}) & \xrightarrow{i_*} & \pi_p(S^n, E_-^n) \end{array}$$

the boundary homomorphism ∂ , and the homomorphism k_* induced by an inclusion map, are isomorphisms onto, and the commutativity relation $i_* = k_* E \partial$ holds, where E is the suspension homomorphism. The statements above are precisely the easier part of the first two theorems of Freudenthal [3].

5. The next problem. Assume that the pair $(X, \mathcal{E}^n)$ is m -connected, $m \geq 1$, as in Theorem 2. Then we know that $\pi_p(X^*; \mathcal{E}^n, X) = 0$ for $p < m + n$. The next problem that arises is to determine the group $\pi_{m+n}(X^*; \mathcal{E}^n, X)$. The dimension $m + n$ will be called the first nontrivial dimension for the triad $(X^*; \mathcal{E}^n, X)$.

A few results have already been obtained in this direction. The Freudenthal suspension theorems, together with some improvements due to G. W. Whitehead [5], show that the group $\pi_{2n-1}(S^n; E_+^n, E_-^n)$ is infinite cyclic if n is odd, and strongly suggest that the same thing is true if n is even. J. H. C. Whitehead [6]

has obtained some theorems which amount to describing the groups and homomorphisms in the following portion of the exact sequence of the pair (X^*, X) :

$$\cdots \xrightarrow{\partial} \pi_n(X) \xrightarrow{i_*} \pi_n(X^*) \xrightarrow{j_*} \pi_n(X^*, X) \xrightarrow{\partial} \pi_{n-1}(X) \rightarrow \cdots$$

Although he was not able to determine the group $\pi_{n+1}(X^*, X)$ in general, his results are closely related to our results on the following part of the exact sequence of the triad $(X^*; \mathcal{E}^n, X)$ in case $m = 1$:

$$\cdots \rightarrow \pi_{n+1}(\mathcal{E}^n, \dot{\mathcal{E}}^n) \rightarrow \pi_{n+1}(X^*, X) \rightarrow \pi_{n+1}(X^*; \mathcal{E}^n, X) \rightarrow \cdots$$

The rest of this paper will be devoted to the description of a general theorem about the homotopy group in the first nontrivial dimension of the triad $(X^*; \mathcal{E}^n, X)$. The proof of the results obtained is much more complicated than the proofs of either Theorems 2 or 3, and cannot be given here. In order to be able to state the main result, it is necessary first to develop several auxiliary concepts.

6. The homotopy groups of a covering. We use the word "covering" in a very special sense: A *covering* of a space X is an *ordered* pair (A, B) of subspaces of X such that $A \cup B = X$ and $A \cap B$ is nonvoid. We denote such a covering by (A/B) .

Let (A/B) be a covering of X in this restricted sense. Choose a base point $x_0 \in A \cap B$. By a *map*, $f: (E_+^n/E_-^n, p_0) \rightarrow (A/B, x_0)$, is meant a continuous function $f: S^n \rightarrow X$ such that $f(E_+^n) \subset A$, $f(E_-^n) \subset B$, and $f(p_0) = x_0$. Homotopy of two such maps is defined in an obvious fashion. Let $\pi_n(A/B, x_0)$ denote the set of all homotopy classes of maps $(E_+^n/E_-^n, p_0) \rightarrow (A/B, x_0)$. If $n > 1$, it is possible to define an addition between the elements of $\pi_n(A/B, x_0)$ in such a way that $\pi_n(A/B, x_0)$ becomes a group, the n th *homotopy group of the covering* (A/B) *at the base point* x_0 . This group is abelian if $n > 2$, but need not be so if $n = 2$. It has many properties which are the analogs of properties of the homotopy groups of pairs and triads.

Associated with a covering (A/B) of X are two exact sequences:

$$\cdots \rightarrow \pi_n(A/B) \rightarrow \pi_n(A, A \cap B) \rightarrow \pi_{n-1}(B) \rightarrow \pi_{n-1}(A/B) \rightarrow \cdots,$$

$$\cdots \rightarrow \pi_n(A/B) \rightarrow \pi_n(B, A \cap B) \rightarrow \pi_{n-1}(A) \rightarrow \pi_{n-1}(A/B) \rightarrow \cdots$$

The homomorphisms in this sequence are defined by means of "boundary operators" and inclusion maps, similar to those in the exact sequence of a pair or a triad.

If $(X; A, B)$ is an arbitrary triad, then the subspaces A and B determine a covering (A/B) of $A \cup B$. The homotopy groups of this covering fit into an exact sequence with the homotopy groups of a triad, as follows:

$$\cdots \rightarrow \pi_{n+1}(X; A, B, x_0) \xrightarrow{\partial} \pi_n(A/B, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X; A, B) \rightarrow \cdots$$

7. Generalized Whitehead products. In [6], J. H. C. Whitehead has defined a "product" which associates with elements $\alpha \in \pi_p(X)$ and $\beta \in \pi_q(X)$ an element $[\alpha, \beta] \in \pi_r(X)$, where $r = p + q - 1$, and proved several properties of this product, the most important of which is that it is bilinear if $p > 1$ and $q > 1$. In an analogous fashion, we can define three new products which are closely related to each other and to the original Whitehead product. One of these products is used in the statement of our main theorem below, and the others are useful in making applications of this theorem.

These three new products are as follows:

(a) Let (X, A) be an arbitrary pair. Then given elements $\alpha \in \pi_p(A, x_0)$ and $\beta \in \pi_q(X, A, x_0)$, the products $[\alpha, \beta]$ and $[\beta, \alpha]$ are elements of $\pi_r(X, A, x_0)$, where $r = p + q - 1$, and $x_0 \in A$ is the base point.

(b) Let (A/B) be a covering of $X = A \cup B$. Then if $\alpha \in \pi_p(A \cap B)$ and $\beta \in \pi_q(A/B)$, the products $[\alpha, \beta]$ and $[\beta, \alpha]$ are defined, and belong to $\pi_r(A/B)$, $r = p + q - 1$.

(c) Let $(X; A, B)$ be an arbitrary triad, $\alpha \in \pi_p(X, A \cap B)$, and $\beta \in \pi_q(A/B)$. Then $[\alpha, \beta]$ and $[\beta, \alpha]$ are elements of $\pi_r(X; A, B)$, $r = p + q - 1$.

All three of these products are bilinear, provided the integers p and q are large enough so that all the groups involved are abelian. All three of them satisfy the following commutation rule:

$$[\alpha, \beta] = (-1)^{pq}[\beta, \alpha].$$

The product mentioned in (a) above is defined as follows. Let

$$f: (E^p, S^{p-1}) \rightarrow (A, x_0),$$

$$g: (E^q, S^{q-1}, E_+^{q-1}) \rightarrow (X, A, x_0)$$

represent $\alpha \in \pi_p(A, x_0)$ and $\beta \in \pi_q(X, A, x_0)$ respectively (assuming, of course, that appropriate orientations have been chosen for the cells E^p and E^q). Let A^r denote the following subset of the product space, $E^p \times E^q$:

$$A^r = (S^{p-1} \times E^q) \cup (E^p \times E_+^{q-1}).$$

Then A^r is a closed r -cell, and its bounding $(r - 1)$ -sphere, $\dot{A}^r$, is given by

$$\dot{A}^r = (S^{p-1} \times E_-^{q-1}) \cup (E^p \times S^{q-2}).$$

Define a map

$$\phi: (A^r, \dot{A}^r) \rightarrow (X, A)$$

by

$$\phi(x, y) = \begin{cases} f(x) & \text{if } (x, y) \in E^p \times E_+^{q-1}, \\ g(y) & \text{if } (x, y) \in S^{p-1} \times E^q. \end{cases}$$

The map ϕ thus defined is continuous, and its homotopy class depends only on the homotopy classes of f and g . The product $[\alpha, \beta]$ is now defined to be the

homotopy class of ϕ ; to complete the definition, it is necessary to make some convention as to choice of a base point in A' and of orientation for the cell A' .

The products mentioned in (b) and (c) above are defined in a similar manner.

As examples of some of the properties of these products, we list the following without proof.

$$(a) \quad \text{Let} \quad \partial_n: \pi_n(X, A, x_0) \rightarrow \pi_{n-1}(A, x_0), \quad n = 2, 3, \dots,$$

denote the homotopy boundary operator of the pair (X, A) , $\alpha \in \pi_p(A, x_0)$, $\beta \in \pi_q(X, A, x_0)$. Then

$$\partial_r[\alpha, \beta] = (-1)^{p-1}[\alpha, \partial_q\beta], \quad \partial_r[\beta, \alpha] = -[\partial_q\beta, \alpha].$$

If $\alpha \in \pi_p(X, A, x_0)$ and $\beta \in \pi_q(X, A, x_0)$, then

$$[\partial_p\alpha, \beta] = (-1)^{p-1}[\alpha, \partial_q\beta].$$

(b) Let $f: (X, A, x_0) \rightarrow (Y, B, y_0)$ be a continuous map, and let

$$f_n: \pi_n(X, A, x_0) \rightarrow \pi_n(Y, B, y_0), \quad n = 2, 3, \dots,$$

$$g_m: \pi_m(A, x_0) \rightarrow \pi_m(B, y_0), \quad m = 2, 3, \dots,$$

be the homomorphisms induced by f . Then

$$f_r[\alpha, \beta] = [g_p\alpha, f_q\beta]$$

for $\alpha \in \pi_p(A, x_0)$, $\beta \in \pi_q(X, A, x_0)$.

(c) Let (X, A) be a pair, and

$$i_m: \pi_m(A, x_0) \rightarrow \pi_m(X, x_0), \quad m = 1, 2, \dots,$$

$$j_n: \pi_n(X, x_0) \rightarrow \pi_n(X, A, x_0), \quad n = 1, 2, \dots,$$

the injections. Then

$$j_r[i_p\alpha, \beta] = [\alpha, j_q\beta]$$

for $\alpha \in \pi_p(A, x_0)$, $\beta \in \pi_q(X, x_0)$.

(d) Let $(X; A, B)$ be an arbitrary triad, and let

$$i: \pi_p(X, A \cap B, x_0) \rightarrow \pi_p(X, A, x_0),$$

$$j: \pi_q(A, x_0) \rightarrow \pi_q(A/B, x_0),$$

$$j': \pi_r(X, A, x_0) \rightarrow \pi_r(X; A, B, x_0),$$

be homomorphisms induced by inclusion maps. Then

$$j'[i\alpha, \beta] = [\alpha, j\beta]$$

for $\alpha \in \pi_p(X, A \cap B, x_0)$, $\beta \in \pi_q(A, x_0)$.

8. The main theorem. Let $X^* = X \cup \mathcal{E}^n$ as in §4. In addition, we make the following assumptions:

- (a) X and $\mathcal{E}^n$ are compact absolute neighborhood retracts.
 - (b) X is simply connected.
 - (c) The maps $\psi_i: (E^n, S^{n-1}) \rightarrow (\mathcal{E}_i^n, \mathcal{E}_i^n)$ map S^{n-1} homeomorphically onto $\mathcal{E}_i^n$.
- Define a homomorphism

$$\Phi: \pi_{m+1}(\mathcal{E}^n/X) \otimes \pi_n(\mathcal{E}^n, \mathcal{E}^n) \rightarrow \pi_{m+n}(X^*; \mathcal{E}^n, X)$$

(where the symbol $\otimes$ signifies "tensor product") as follows: If $\alpha \in \pi_{m+1}(X/\mathcal{E}^n)$ and $\beta \in \pi_n(\mathcal{E}^n, \mathcal{E}^n)$, then

$$\Phi(\alpha \otimes \beta) = [\alpha, i\beta]$$

where $i: \pi_n(\mathcal{E}^n, \mathcal{E}^n) \rightarrow \pi_n(X^*, \mathcal{E}^n)$ is the injection and the brackets denote a generalized Whitehead product. The fact that Φ is a homomorphism follows from the bilinearity of this Whitehead product. We can now state our main theorem, as follows:

THEOREM 4. *If the pair $(X, \mathcal{E}^n)$ is m -connected, $m \geq 1$, then Φ is an isomorphism onto.*

Note that under the assumptions we have made, $\pi_n(\mathcal{E}^n, \mathcal{E}^n)$ and $\pi_{m+1}(\mathcal{E}^n/X)$ are isomorphic to the integral homology groups $H_n(\mathcal{E}^n, \mathcal{E}^n)$ and $H_{m+1}(X, \mathcal{E}^n)$ respectively, both of which are effectively calculable in case the pair $(X, \mathcal{E}^n)$ is triangulable. Hence the group $\pi_{m+n}(X^*; \mathcal{E}^n, X)$ is also effectively calculable in this case.

The proof of this theorem is much more difficult than the proof of Theorems 2 and 3. The only proof that the author knows requires the use of the Steenrod functional cup product (see [4]), although the functional cup product is not used in the statement of the theorem itself. The functional cup product gives an effective method of determining whether or not two maps

$$(E^{m+n}; E_+^{m+n-1}, E_-^{m+n-1}) \rightarrow (X^*; \mathcal{E}^n, X)$$

represent the same element of $\pi_{m+n}(X^*; \mathcal{E}^n, X)$.

9. Applications of the main theorem. If Theorem 4 is applied to the triad $(S^n; E_+^n, E_-^n)$, we find that $\pi_{2n-1}(S^n; E_+^n, E_-^n)$ is infinite cyclic; a generator of this group is obtained by forming the generalized Whitehead product $[\alpha, \beta]$, where α is a generator of the infinite cyclic group $\pi_n(E_+^n/E_-^n)$, and β is one of the generators of the group $\pi_n(S^n, S^{n-1})$, corresponding either to a generator of $\pi_n(E_+^n, S^{n-1})$, or to a generator of $\pi_n(E_-^n, S^{n-1})$. This fact, together with the relations between the various generalized Whitehead products, and elementary properties of the Whitehead products for the homotopy groups of spheres, gives the Freudenthal theorems in the critical dimensions, together with the improvements due to G. W. Whitehead [5] referred to previously.

As a second application, assume that $X^* = X \cup \mathcal{E}^n$ as in §4, and that conditions (a) and (b) of §8 are satisfied; we do *not* assume that (c) of §8 is satis-

fied. We assume that X is m -connected, $m \geq 1$, but $\pi_{m+1}(X) \neq 0$, and wish to determine as many of the groups $\pi_p(X^*, X)$ as possible. In case $m \geq n - 1$, then all the maps $\psi'_i: S^{n-1} \rightarrow X$ (defined by the maps $\psi_i: (E^n, S^{n-1}) \rightarrow (\xi_i^n, \xi_i^n)$) are inessential, and we may as well assume that ξ^n consists of a single point. This case is best handled by other methods; hence we assume that $m < n - 1$. Furthermore, we shall assume that $\psi'_i(p_0) = x_0 \in X$, a fixed base point, for $i = 1, \dots, k$.

Let $(E_1^n, S_1^{n-1}), \dots, (E_k^n, S_k^{n-1})$ be k distinct homeomorphic copies of the pair (E^n, S^{n-1}) , and let

$$\xi_i: (E_i^n, S_i^{n-1}) \rightarrow (E^n, S^{n-1}), \quad i = 1, \dots, k,$$

be homeomorphisms. Let $p_i = \xi_i^{-1}(p_0)$, $i = 1, \dots, k$. Identify the points $p_1, p_2, \dots, p_k$ to a single point $\bar{p}_0$, and denote the identification space thus obtained by E ; then $E = E_1^n \cup \dots \cup E_k^n$. Let $S = S_1^{n-1} \cup \dots \cup S_k^{n-1}$. Define a map

$$\psi: (E, S) \rightarrow (X^*, X)$$

by $\psi|E_i^n = \psi_i \xi_i$. The map ψ induces homomorphisms

$$\psi_*: \pi_p(E, S) \rightarrow \pi_p(X^*, X).$$

We are now in a position to state our results on the groups $\pi_p(X^*, X)$. The problem breaks up into three cases.

(a) *Case where $m = n - 2$.* In this case the homomorphism $\psi_*: \pi_p(E, S) \rightarrow \pi_p(X^*, X)$ is an isomorphism onto for $p < m + n$, and is an isomorphism into for $p < 2n - 2$. Define a homomorphism

$$\zeta: \pi_{m+1}(X) \otimes \pi_n(X^*, X) \rightarrow \pi_{m+n}(X^*, X)$$

by $\zeta(\alpha \otimes \beta) = [\alpha, \beta]$ (Whitehead product). Then ζ is an isomorphism into, and $\pi_{m+n}(X^*, X)$ is the direct sum of the subgroups $\psi_*\pi_{m+n}(E, S)$ and $\zeta[\pi_{m+1}(X) \otimes \pi_n(X^*, X)]$.

(b) *Case where $m = n - 2$ (first possibility).* The map $\psi'_i: (S^{n-1}, p_0) \rightarrow (X, x_0)$ represents an element $\alpha_i \in \pi_{n-1}(X, x_0)$, $i = 1, \dots, k$. Assume first that the elements $\alpha_1, \dots, \alpha_k$ generate the entire group $\pi_{n-1}(X, x_0)$. Then the homomorphism $\psi_*: \pi_p(E, S) \rightarrow \pi_p(X^*, X)$ is an isomorphism onto for $p < 2n - 2$ and a homomorphism onto for $p = 2n - 2$. The kernel of the homomorphism $\psi_*: \pi_{2n-2}(E, S) \rightarrow \pi_{2n-2}(X^*, X)$ is the subgroup of $\pi_{2n-2}(E, S)$ generated by the elements $[\alpha, \beta]$, where α is an arbitrary element of $\pi_{n-1}(S)$, and β is an element of $\pi_n(E, S)$ which satisfies the condition $\partial\psi_*(\beta) = 0$, where $\partial: \pi_n(X^*, X) \rightarrow \pi_{n-1}(X)$ is the boundary operator of the pair (X^*, X) .

(c) *Case where $m = n - 2$ (second possibility).* Finally, we assume that the elements $\alpha_1, \dots, \alpha_k$ generate a proper subgroup of $\pi_{n-1}(X, x_0)$. Then the homomorphism $\psi_*: \pi_p(E, S) \rightarrow \pi_p(X^*, X)$ is an isomorphism onto for $p < 2n - 2$,

but need not be an isomorphism, nor onto for $p = 2n - 2$. One can assert, however, that the group $\pi_{2n-2}(X^*, X)$ is generated by the elements $\psi_*(\alpha)$ for $\alpha \in \pi_{2n-2}(E, S)$, and the Whitehead products $[\beta, \gamma]$, for $\beta \in \pi_{n-1}(X)$, $\gamma \in \pi_n(X^*, X)$.

These results are all a direct application of Theorem 4, together with the basic properties of the Whitehead products.

Another application of these results is to the study of the homotopy groups of a complex which is a union of spheres with a single point in common. Let S_k^n be a cell complex which has a single vertex, e^0 , and k n -dimensional cells, $e_1^n, \dots, e_k^n$ ($n > 1$). The closure of the cell e_i^n is an n -sphere, denoted by S_i^n . The space S_k^n is the union of the spheres $S_1^n, \dots, S_k^n$, and $S_i^n \cap S_j^n = e^0$ if $i \neq j$. Let $\phi_i: S_i^n \rightarrow S_k^n$ be an inclusion map; ϕ_i represents an element $\alpha_i \in \pi_n(S_k^n)$; the elements $\alpha_1, \dots, \alpha_k$ are a set of free generators of the group $\pi_n(S_k^n)$. Let $\phi_{ij}: S^{2n-1} \rightarrow S_k^n$ be a map representing the Whitehead product $[\alpha_i, \alpha_j]$. Then it can be shown that $\phi_{i*}: \pi_p(S_i^n) \rightarrow \pi_p(S_k^n)$ is an isomorphism into in all dimensions; and $\phi_{ij*}: \pi_p(S^{2n-1}) \rightarrow \pi_p(S_k^n)$ is an isomorphism into if $p < 4n - 4$ and $i \neq j$. If $p < 3n - 2$, then the group $\pi_p(S_k^n)$ is the direct sum of the subgroups $\phi_{i*}\pi_p(S_i^n)$, $i = 1, \dots, k$, and $\phi_{ij*}\pi_p(S^{2n-1})$, $1 \leq i < j \leq k$. These results are most easily proved by applying a theorem of G. W. Whitehead ([5], Theorem 4.8) on the homotopy groups of the union of two spaces with a single point in common. This theorem enables one to prove these results on $\pi_p(S_k^n)$ by means of an induction on k , the number of spheres. It is also necessary to use Theorems 2 and 3 in the process.

By using this same method, and applying Theorem 4, it is possible to determine the structure of $\pi_{3n-2}(S_k^n)$, as follows:

THEOREM 5. *The group $\pi_{3n-2}(S_k^n)$ is the direct sum of the following subgroups:*

- (1) $\phi_{i*}\pi_{3n-2}(S_i^n)$, $1 \leq i \leq k$,
- (2) $\phi_{ij*}\pi_{3n-2}(S^{2n-1})$, $1 \leq i < j \leq k$,
- (3) *a free abelian group on $\frac{1}{3}(k+1)k(k-1)$ generators, generated by the triple Whitehead products*

$$[\alpha_i, [\alpha_j, \alpha_l]], \quad 1 \leq j < l \leq k, \quad 1 \leq i \leq l.$$

Rather than apply Theorem 4 directly to prove Theorem 5, it is more convenient to use the result stated in (a) of this section.

10. Unsolved problems. The major unsolved problem in this subject at present is the determination of the homotopy groups of the triad $(X^*; \mathcal{E}^n, X)$ in higher dimensions. The method used to determine the triad homotopy groups in the first nontrivial dimension breaks down completely when applied to higher dimensions. Apparently this problem is a major obstacle to the further progress of homotopy theory in general. Even the groups $\pi_{2n}(S^n; E_+^n, E_-^n)$ are unknown, except in the cases $n = 2, 4$, and 8 , in which case $\pi_{2n}(S^n; E_+^n, E_-^n) \approx \pi_{2n}(S^{2n-1})$.

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PRINCETON UNIVERSITY,
PRINCETON, N. J., U. S. A.

HOMOLOGY INVARIANTS AND FIBRE BUNDLES

GUY C. HIRSCH

1. Homology invariants¹ can be associated with fibre bundles (written f.b.)² in two main ways. Namely, (a) one can consider the relations between *homology properties* of the base space M , the bundle $\tilde{M}$, and the fibre $\tilde{F}$; or (b) one can also investigate a *classification* of f.b. based on homology properties only and, for instance, determine some of the classical invariants (characteristic class, obstructions, ...) by the consideration of homology properties, or vice-versa. We shall be concerned here mainly with the links between problems (a) and (b).

2. Problem (a) can be stated as follows:³ What relations can be established between the cohomology rings (or even their additive structure) $\tilde{H}$, H , $\tilde{H}$ respectively of the bundle $\tilde{M}$, the base space M , and the fibre $\tilde{F}$, or even between their rings of cochains $\tilde{C}$, C , $\tilde{C}$?

Several contributions to the solution of this very general problem were given by Leray [18], particularly relations between the Poincaré polynomials of the three spaces and even part of the ring structure. But for our purposes it will be convenient to consider separately various aspects of this problem (a):

a1 (direct problem): Given M and $\tilde{F}$, what can be said about the homology of $\tilde{M}$?

a2 (inverse problem): Given $\tilde{M}$ and $\tilde{F}$, what can be said about the homology of M ?

and even a3 (compatibility problem): Given $\tilde{M}$, what is the homology type of fibres $\tilde{F}$ compatible with the existence of a fibre structure in $\tilde{M}$?

We shall be concerned mainly with problem a1 and its implications (in §§6 ff.).

3. Problem a1 was solved in 1941 for the additive structure of $\tilde{H}$ and for particular spaces, firstly [8] when $\tilde{F}$ is an S^0 (2-sheeted covering) or a circle S^1 , and [6] when (all spaces considered being manifolds) $\tilde{F}$ is a (homology) n -dimensional sphere S^n . In 1942, Samelson [19] determined $\tilde{H}$ when the base space M is a sphere S^n and a homogeneous space ($\tilde{M}$ and $\tilde{F}$ being group manifolds), and found two different types of spaces. Although as a rule I shall not consider in this paper properties which depend on more than homology properties, I mention Samelson's results because it can be shown⁴ that they depend only on the nature of the cohomology rings of $\tilde{M}$ and $\tilde{F}$ (which are isomorphic to the rings

¹ Here I shall generally consider singular homology in locally compact spaces.

² No structure groups will be considered except in §15.

³ The similar problems for U , and the reduced powers of Steenrod are not considered here.

⁴ This is a (yet unpublished) consequence of the general formulae mentioned in §8 and §10.

of topological products of spheres), $\tilde{M}$ being only a homology sphere. When the ring of $\tilde{M}$ has not this property, there is a third type of spaces.⁵

The first general formulae were given in 1946 by Leray [18].⁶

4. Problem a2 was solved by Gysin [6] when $\tilde{M}$ and $\tilde{F}$ are (homology) spheres.

A particular case of a2 is given by *homogeneous spaces* ($\tilde{M}$ and $\tilde{F}$ being group manifolds). Important results were obtained by Leray [18], Koszul [17], H. Cartan [4]. These results do *not* hold if $\tilde{M}$ and $\tilde{F}$ have only the (additive) homology groups of group manifolds.

H. Cartan [3] solved a problem similar to a2, $\tilde{M}$ being a space in which a group operates.

5. Problem a3 is far from being solved in the general case. Some recent results are known when $\tilde{M}$ is a euclidean space [1]: the fibre (if different from a point) is never compact; or when $\tilde{M}$ is a sphere: $\tilde{F}$ cannot be a topological product of m circles ($m > 1$) [5]. Although the proof of the latter theorem uses homotopy methods, I believe that the theorem actually belongs to the theory of homology; it is, for instance, easy to prove (as a consequence of the general results of §8 and §10) that a homology sphere cannot be a f.b. where the fibre has the same cohomology ring as the product of two spheres (of arbitrary dimension). It might even be possible to prove that a (homology) sphere cannot be a f.b. where $\tilde{F}$ is neither a homology sphere nor a homology simplex. *Added in proof:* A. Borel proved recently (C. R. Acad. Sci. Paris vol. 231 (1950) pp. 943–945) that a homology sphere cannot be a f.b. where the fibre has the same cohomology ring as a product of m spheres ($m > 1$).

6. Although problem a1 finds an important application to *coverings* (given the base space M and the Poincaré group $\tilde{\pi}_1$ of the covering space $\tilde{M}$ as a subgroup of the Poincaré group π_1 of M), we shall restrict ourselves to the case where the domain of coefficients is a *field* and where π_1 operates trivially on the cohomology of $\tilde{F}$ (in the given field of coefficients).

Before entering into details about problem a1 and its relation to problem (b), I would like to mention two more problems which are closely related to problem (a), and might be called its “dynamic” aspect:

Problem (c): Let $\tilde{T}$ be a *mapping* of a f.b. $\tilde{M}$ in a f.b. $\tilde{M}'$, compatible with the fibre structure (this means that $\tilde{T}$ maps all the points belonging to the same fibre $\tilde{F}$ of $\tilde{M}$ into points belonging to the same fibre $\tilde{F}'$ of $\tilde{M}'$). The homology type of such a mapping is determined to a certain extent (as can be proved by applying the general theory of §8 and §10 to the induced mappings T of the

⁵ In 1949, Wang [22] gave some results for f.b. where the base space is a homology sphere, under certain assumptions. The latter lead him back to the two types considered by Samelson.

⁶ Leray considered even the more general problem of mappings instead of fibre bundles.

base space M into the base space M' , and $\tilde{T}$ of $\tilde{F}$ into $\tilde{F}'$ —the latter being not unique, but having a constant homology type). Examples of this kind (with $\tilde{M} = \tilde{M}'$) are Borsuk's theorem [2] (1933) on “antipodentreue Abbildungen” (i.e., $\tilde{F}$ being a pair of antipodic points, $\tilde{M}$ the sphere and M the projective space, the degree of $\tilde{T}$ is *odd* when $\tilde{F}$ is mapped *onto* a fibre), or Hopf-Rueff's theorem [15] (1938), where $\tilde{M}$ is an S^{2k+1} , $\tilde{F}$ a circle, and M a k -dimensional complex projective space: the degree of $\tilde{T}$ is an exact $(k+1)$ th power [the $(k+1)$ th power of the degree of the mapping $\tilde{T}$ induced in $\tilde{F}$]. Generalisations of these theorems were given [9] when $\tilde{F}$ is discrete, or [8] when $\tilde{F}$ is a circle, $\tilde{M}$ being an arbitrary polyhedron (see also [7]).

Problem (d) is the following: Determine the homology type of a mapping T of a space M into a space M' , such that it induces a mapping $\tilde{T}$ of a f.b. $\tilde{M}$ over M as base space into a f.b. $\tilde{M}'$ over M' as base space (for instance, $\tilde{M}$ being a covering of M). Example: M being a projective space and T a mapping of M into itself, the knowledge of the homomorphism (induced by T) in the Poincaré group π_1 determines the parity of the degree of T (or its absolute degree) [7].

7. In problem a1, it is of course not sufficient to know properties of M and $\tilde{F}$ to determine the cohomology ring $\tilde{\mathfrak{S}}$ of $\tilde{M}$, as the f.b. $\tilde{M}$ itself is in general not uniquely determined when M and $\tilde{F}$ are given. Some invariants will be needed to describe the type of f.b. considered. It will be convenient to express these invariants in terms of a relation between $\tilde{F}$ and M , which will allow the determination of $\tilde{\mathfrak{S}}$. As the classification of f.b. also rests usually on relations between $\tilde{F}$ and M (e.g., characteristic classes or obstructions), relations between both can be investigated. I showed in 1941 [8], that when $\tilde{F}$ is S^0 or S^1 (the result even applying to a fibre S^n , although this was not then considered in my paper), the invariant describing the homology structure of $\tilde{M}$ is equivalent to the characteristic class defined by Whitney [23]. This result is a special case of a theorem mentioned below (§9), connecting the “reduced characteristic isomorphism” and Steenrod's characteristic cocycle.

8. In Leray's solution [20] of problem a1, a sequence of differential operators is constructed, starting in the tensor product $\mathfrak{S} \otimes \mathfrak{S}$, and yielding, after a finite number of steps, a *filtration* of the cohomology ring $\tilde{\mathfrak{S}}$ (each ring being the “cohomology ring” of the preceding one; the sequence is a Leray-Koszul sequence [16]). Leray's operators use only homology properties, and are a topological invariant of the f.b. In my approach to the problem [10], a unique differential operator in the tensor product $\mathfrak{C} \otimes \mathfrak{S}$ ($\mathfrak{C}$ being the ring of cochains) yields the additive structure and part of the multiplicative structure of $\tilde{\mathfrak{S}}$. But this differential operator is, in general, not uniquely determined, as its definition rests on the “characteristic isomorphism” of $\tilde{\mathfrak{S}}$ in $\mathfrak{S}$ or $\mathfrak{S} \otimes \mathfrak{S}$, which depends on the choice of representatives, in the ring of cochains of $\tilde{M}$, of elements of cohomology classes of $\tilde{F}$.

It is possible to construct for $\check{S}$ a composition series without repeated terms,

$$0 \subset \check{S}_0 \subset \cdots \subset \check{S}_i \subset \cdots \subset \check{S}_s \subset \check{S}_{s+1} = \check{S};$$

$\check{S}_0$ is in $\check{S}$ the subgroup of images of $\check{S}$.

Let $\mathcal{C}$ and $\tilde{\mathcal{C}}$ be respectively the rings of cochains of M and $\tilde{M}$; let P be the homomorphism of $\mathcal{C}$ in $\tilde{\mathcal{C}}$. Calling $\check{z}_i$ an element of $\check{S}_i/\check{S}_{i-1}$ and $\widetilde{\text{ext}}(\check{z}_i)$ a cochain in $\tilde{M}$ whose restriction to any fibre belongs to the class $\check{z}_i$, and defining a homomorphism P_i of $\mathcal{C} \otimes \check{S}_i/\check{S}_{i-1}$ in $\tilde{\mathcal{C}}$ by $P_i(C \otimes \check{z}_i) = P(C) \cup \widetilde{\text{ext}}(\check{z}_i)$, $\check{S}_{i+1}$ is then defined by the property

$$\delta[\widetilde{\text{ext}}(\check{z}_{i+1})] \in P_i(\mathcal{C} \otimes \check{S}_i/\check{S}_{i-1}).$$

The choice of a representative $\widetilde{\text{ext}}(\check{z}_{i+1})$ associates with each class $\check{z}_{i+1}$, by this definition, an element of $\mathcal{C} \otimes \check{S}_i/\check{S}_{i-1}$, and even an element of $\check{S} \otimes \check{S}_i/\check{S}_{i-1}$. I call this correspondence the *characteristic isomorphism* W of $\check{S}_{i+1}$ in $\check{S} \otimes \check{S}_i/\check{S}_{i-1}$.⁷ Its definition depends essentially on the choice of $\widetilde{\text{ext}}$.⁸

However, defining $\check{S}_* \subset \check{S}$ and $\check{z}_* \in \check{S}_*$ by $\delta[\widetilde{\text{ext}}(\check{z}_*)] \in P(\mathcal{C})$, then this relation defines an *invariant isomorphism* W_* (called *reduced characteristic isomorphism*) of $\check{S}_*/\check{S}_0$ on a factor group of a subgroup of $\check{S}$.⁹

9. For the smallest nontrivial dimension h in $\check{S}$, the reduced characteristic isomorphism maps on a subgroup of $\check{S}$ and is equivalent¹⁰ to Steenrod's characteristic cocycle [20], as can be shown by the values taken on an element of $\check{S}$ by the latter applied to a cycle of M .

10. The characteristic isomorphism is sufficient for determining the (additive) group $\check{S}$ (and even part of the multiplicative structure). P_i induces an isomorphism of $\mathfrak{F}_i^*/\mathfrak{R}_i^*$ (both subgroups of $\check{S} \otimes \check{S}_i/\check{S}_{i-1}$) on $\check{S}_i/\check{S}_{i-1}$, where $\check{S}_i$ is an element of a composition series

$$\check{S} = \check{S}_s \supset \check{S}_{s-1} \supset \cdots \supset \check{S}_i \supset \cdots \supset \check{S}_0 \supset 0.¹¹$$

Define a product $\smile$ between $\mathcal{C}$ and $\mathcal{C} \otimes \check{S}_i/\check{S}_{i-1}$ by the rule $C \smile (C' \otimes \check{z}_i) = (C \cup C') \otimes \check{z}_i$; define a homomorphism D_i mapping $\mathcal{C} \otimes \check{S}_i/\check{S}_{i-1}$ into $\mathcal{C} \otimes \check{S}_{i-1}/\check{S}_{i-2}$ by the rule $D_i(C \otimes \check{z}_i) = (-1)^a C \smile W(\check{z}_i)$ where $a = \dim C \cdot \dim \check{z}_i$. Define $\mathfrak{Q}_i^* \subset \mathcal{C} \otimes \check{S}_i/\check{S}_{i-1}$ by $\delta \mathfrak{Q}_i^* \subset D_{i+1} \mathfrak{Q}_{i+1}^*$, and $\mathfrak{M}_i^* \subset \mathcal{C} \otimes \check{S}_i/\check{S}_{i-1}$ by $D_i \mathfrak{M}_i^* \subset \delta \mathfrak{M}_{i-1}^*$.

Then $\mathfrak{F}_i^*$ is the group of classes of $\mathfrak{Z} \otimes \check{S}_i/\check{S}_{i-1}$ contained in $\mathfrak{M}_i^*$ (where $\mathfrak{Z}$ is

⁷ This is essentially the process defined in [10], where, however, I considered simultaneously homology and cohomology, which made the definitions more cumbersome.

⁸ Even the length s of the series is not invariant.

⁹ A similar notion has been introduced by Koszul [17] for homogeneous spaces, under the name of *transgression*.

¹⁰ See the condition about π_1 in §6.

¹¹ For the intuitive aspect of this homomorphism, see [11].

the group of cocycles in M), and $\mathfrak{R}_i^*$ is the group of classes of $\mathfrak{Z} \otimes \check{\mathfrak{S}}_i / \check{\mathfrak{S}}_{i-1}$ containing $D_{i+1} \mathfrak{Q}_{i+1}^*$.¹²

11. Lack of space prevents me from enumerating properties of W ; $\check{\mathfrak{S}} / \check{\mathfrak{S}}_0$ is similar in many respects to an exterior algebra, elements of $\check{\mathfrak{S}} / \check{\mathfrak{S}}_0$ being maximal in the ring and of odd dimension. But although the factor ring $\check{\mathfrak{S}} / \check{\mathfrak{S}}_0$ is generated by the elements appearing in the reduced characteristic isomorphism when M is a homogeneous space [17], [4], a similar result does *not* apply to general f.b. Also the elements $W_*(\check{\mathfrak{S}}_* / \check{\mathfrak{S}}_0)$ which are generators of $\check{\mathfrak{S}}$ for homogeneous spaces are not maximal, but only relatively maximal, in the general case.

12. The system of all characteristic isomorphisms (for all possible choices of representatives) is of course an invariant of the f.b. The relations between it and Leray's invariants might be investigated; it is probable that both are equivalent.

13. Neither the characteristic isomorphism nor Leray's sequence are sufficient to determine fully the multiplicative structure of $\check{\mathfrak{S}}$. It follows from §10 that the latter depends on the products of the extensions $\text{ext}(\check{z}_i)$. These products define new invariants which attach to pairs of elements of $\check{\mathfrak{S}}$ elements of $\check{\mathfrak{S}} \otimes \check{\mathfrak{S}}$ when $\check{z}$ is a $\check{z}_0$ belonging to $\check{\mathfrak{S}}_0$, or homomorphisms in these tensor products for other $\check{z}_i$ [12].

14. When the fibre is an S^2 or, more generally, a k -dimensional complex projective space, these invariants are the same as those introduced by Hopf and relative to the second obstruction [14] [13]. This can be proved by the construction of a new f.b. $\tilde{M}$ over M , replacing $\check{F}$ by an S^3 (or an S^{2k+1}) which is itself a f.b. over $\check{F}$, and using the definition of the characteristic class mentioned in §9 (as Hopf's second obstruction of the first f.b. becomes the characteristic class of the new f.b.). It is still an open problem to relate the invariants of the multiplicative structure to known invariants of a f.b. when $\check{F}$ is not a projective space.

15. The latter considerations will lead to a new point of view. In this section, I shall consider a f.b. with a structure group $\mathfrak{G}$ which has an invariant subgroup $\mathfrak{G}_0$ and operates transitively on $\check{F}$. $\check{F}$ is itself a f.b. with a fibre $\check{F}_0$ (defined by equivalence relative to $\mathfrak{G}_0$), and F as base space. In the given f.b. $\tilde{M}$, consideration of the equivalence classes relative to $\mathfrak{G}_0$ leads to a new f.b. M , which is

¹² An important special case ("reducible case") allows $\mathfrak{Q}_i^*$ to be replaced by $\mathfrak{Z} \otimes \check{\mathfrak{S}}_i / \check{\mathfrak{S}}_{i-1}$ and $\mathfrak{M}_i^*$ by $\mathfrak{C} \otimes \check{\mathfrak{S}}_i / \check{\mathfrak{S}}_{i-1}$, thus considering D as a derivation in the direct sum of $\check{\mathfrak{S}} \otimes \check{\mathfrak{S}}_i / \check{\mathfrak{S}}_{i-1}$ (graduated by the index i) for which the factor groups $\mathfrak{F}_i^* / \mathfrak{R}_i^*$ appear as homology groups (with D as boundary operator). It was, for instance, proved by Koszul that the reducible case applies to symmetric homogeneous spaces.

the base space for the f.b. $\tilde{M}$ (with the fibre $\tilde{F}_0$), and also a f.b. over M with F as fibre. (Example, similar to the last part of §14: $\tilde{F}$ being an S^{2k+1} , $\mathcal{G}$ the unimodular group, and $\mathcal{G}_0$ the subgroup leaving one point invariant; F is a complex projective space, $\tilde{F}_0$ a circle. In the problem of §14, it was M that was the given f.b., and $\tilde{M}$ the auxiliary one. A similar theory holds mod 2 with $\mathcal{G}$ the orthogonal group, F being a real projective space.)

The invariants of the multiplicative structure of the auxiliary f.b. $\tilde{M}$ (defined as in §13) are invariants of the f.b. M . For instance, when $\tilde{F}$ is a sphere and $\mathcal{G}$ the orthogonal or the unimodular group, these invariants coincide with the characteristic classes of Stiefel-Whitney or Chern [13], which are thus defined without explicitly considering associated f.b.

16. In this respect, I might also mention here recent results of Thom [21], who determined these characteristic classes when $\tilde{F}$ is a sphere, by means of the reduced powers of Steenrod (thus proving their topological invariance and independence of the differential structure). However, these interesting results do not come quite in the scope of the present paper, as they would not apply if $\tilde{F}$ were only a *homology* sphere.

17. When the f.b. is the topological product of M and $\tilde{F}$, it is known that a *section* is equivalent to a *mapping* of M in $\tilde{F}$, thus defining a *homomorphism* (Hopf's homomorphism) of the cohomology group $\tilde{\mathfrak{H}}$ of $\tilde{F}$ in the cohomology group $\mathfrak{H}$ of M . It can be shown [13]¹³ that (when one section is given), a section in a general f.b. (which does not define a mapping) also determines a homomorphism of $\tilde{\mathfrak{H}}_0$ (defined in §8) in $\mathfrak{H}$; for elements of $\tilde{\mathfrak{H}}_i$ ($i \neq 0$), there corresponds a cochain in M , or a homomorphism of $\tilde{\mathfrak{H}}_i^*$ (definition, §10) in $\mathfrak{H}$.

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¹³ This was proved as a reply to a question by Hopf.

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UNIVERSITY OF BRUSSELS,
BRUSSELS, BELGIUM.

HOMOLOGY THEORY OF FIBER BUNDLES

E. H. SPANIER

1. The first comprehensive result on the homology properties of fiber bundles was obtained by Gysin [3] who studied sphere bundles over manifolds. Little more was done with this subject until fairly recently when several people began to get interested in it. Among these should be mentioned Leray [7, 8], Hirsch [4, 5, 6], Steenrod [12], Thom [13, 14], Wu [17], Borel and Serre [1], and Shapiro [9]. The chief tool in the approach of Leray and many of the others seems to be the "ring of the projection" of the bundle. This is an algebraic structure associated with the bundle which is obtained by considering cohomology of the base space with local coefficients in the cohomology ring of the fiber. In the general situation this algebraic structure is fairly complicated, and it is difficult to compute with it.

A method similar to that of Leray has been utilized by Chern and the author [2] to study the homology structure of fiber bundles where the base space is assumed to be a complex. The technique consists of a study of the homology properties of the inverse images of the skeletons of the base space under the projection. The resulting algebraic structure is essentially the same as the ring of the projection; however, because of the assumption that the base space is a complex, the situation is simpler than the general case and computations can be carried out more easily.

In this paper this algebraic machinery is discussed and some applications are given. In particular, it is shown how the characteristic class of the bundle can be obtained rather simply in this approach. By considering the graph of the projection as a bundle over the base space, some of the multiplicative properties of the cohomology ring of the bundle can be obtained. In this way a fairly simple theorem on sphere bundles is obtained which includes the main results of Gysin. The last application discussed is to the relative position of the fiber in the bundle. If the fiber is totally nonhomologous to zero, it is shown that the cohomology groups of the bundle are isomorphic to the corresponding groups of the product space of the base space by the fiber.

The algebraic structure which arises in this study seems to be of interest in other connections. Any simplicial map of one complex into another gives rise to a similar situation, and an investigation of this would perhaps lead to new invariants of mappings which could be applied to the homotopy classification problem. Another situation in which this algebraic structure is of interest is obtained by considering the homotopy groups of the skeletons of a complex. In this case the new sequence of J. H. C. Whitehead [16], can be obtained by this approach, and perhaps interesting generalizations of this can be found by deeper inspection.

2. Let $f: B \rightarrow X$ be a fibering with connected fiber F and structure group Γ [11]. We shall assume X is a finite connected cell complex of dimension n in which each cell is simply connected. We also assume that if σ is any cell of X , there is a homeomorphism

$$\phi_\sigma: |\sigma| \times F \rightarrow f^{-1}(|\sigma|)$$

such that

$$f\phi_\sigma(x, y) = x$$

for $x \in |\sigma|$ and $y \in F$. Moreover, if σ is a face of τ , the transformation

$$\gamma_{\sigma\tau}(x): F \rightarrow F$$

defined for $x \in |\sigma|$ by

$$\phi_\sigma(x, y) = \Phi_\tau(x, \gamma_{\sigma\tau}(x)(y))$$

is an element of Γ depending continuously on x in $|\sigma|$.

Let X^r denote the r -dimensional skeleton of X and set $B_r = f^{-1}(X^r)$. As coefficient system for cohomology in B we use a local system $\{G_b\}$ [10] which is simple over every fiber. Such a system is the inverse image under f of some local system $\{G_x\}$ on X . Since $|\sigma|$ is simply connected, $\{G_x\}$ can be replaced by a local system $\{G_\sigma\}$ over the cells of X .

From the fact that $f^{-1}(|\sigma|)$ is homeomorphic to $|\sigma| \times F$, an isomorphism

$$\lambda: H^p(B_q, B_{q-1}; G_b) \approx C^q(X; H^{p-q}(F_\sigma; G_\sigma))$$

can be constructed in a natural fashion. In the above the first group is the p -dimensional relative cohomology group of B_q modulo B_{q-1} with coefficients in $\{G_b\}$, while the second group is the group of q -dimensional cochains of X with coefficients in the $(p - q)$ -dimensional cohomology group of F_σ with coefficients in G_σ . Furthermore, if

$$j: B_q \subset (B_q, B_{q-1})$$

denotes the inclusion map and δ^* denotes the coboundary operator for the pair (B_{q+1}, B_q) , commutativity holds in the diagram

$$\begin{array}{ccccc} H^p(B_q, B_{q-1}; G_b) & \xrightarrow{j^*} & H^p(B_q; G_b) & \xrightarrow{\delta^*} & H^{p+1}(B_{q+1}, B_q; G_b) \\ \lambda \downarrow & & & & \downarrow \lambda \\ C^q(X; H^{p-q}(F_\sigma; G_\sigma)) & \xrightarrow{\delta} & C^{q+1}(X; H^{p-q}(F_\sigma; G_\sigma)) & & \end{array}$$

where δ denotes the coboundary map for cochains of X .

The following diagram will be called the *main diagram of the bundle*.

$$\begin{array}{ccccccc}
 & & H^p(B_n, B_{n-1}; G_b) & \xrightarrow{j^*} & H^p(B_n = B; G_b) & & \\
 & & \downarrow i^* & & & & \\
 \cdots & \xrightarrow{\delta^*} & H^p(B_{n-1}, B_{n-2}; G_b) & \xrightarrow{j^*} & H^p(B_{n-1}; G_b) & \xrightarrow{\delta^*} & H^{p+1}(B_n, B_{n-1}; G_b) \xrightarrow{j^*} \cdots \\
 & & \downarrow i^* & & & & \\
 & & \vdots & & & & \\
 & & \downarrow i^* & & & & \\
 \cdots & \xrightarrow{\delta^*} & H^p(B_p, B_{p-1}; G_b) & \xrightarrow{j^*} & H^p(B_p; G_b) & \xrightarrow{\delta^*} & H^{p+1}(B_{p+1}, B_p; G_b) \xrightarrow{j^*} \cdots \\
 & & \downarrow i^* & & & & \\
 & & \vdots & & & & \\
 & & \downarrow i^* & & & & \\
 \cdots & \xrightarrow{\delta^*} & H^p(B_0, B_{-1}; G_b) & \xrightarrow{j^*} & H^p(B_0; G_b) & \xrightarrow{\delta^*} & H^{p+1}(B_1, B_0; G_b) \xrightarrow{j^*} \cdots \\
 & & \downarrow i^* & & & & \\
 & & 0. & & & &
 \end{array}$$

Note that any sequence composed of j^* , i^* , δ^* in that cyclic order is exact.

Let $u \in H^p(B_q, B_{q-1}; G_b)$. We say that u can be lifted k -stages to $\bar{u} \in H^{p+1}(B_{q+k+1}, B_{q+k}; G_b)$ if there is $v \in H^p(B_{q+k}; G_b)$ such that $(i^*)^k v = j^* u$ and $\delta^* v = \bar{u}$. Naturally it can happen that a given $u \in H^p(B_q, B_{q-1}; G_b)$ can perhaps be lifted k -stages to different elements of $H^{p+1}(B_{q+k+1}, B_{q+k}; G_b)$. From the exactness properties of the main diagram it is clear that a given $u \in H^p(B_q, B_{q-1}; G_b)$ can be lifted 1-stage if and only if λu is a cocycle of $C^q(X; H^{p-q}(F_\sigma; G_\sigma))$.

3. Let G be an abstract group isomorphic with G_σ . Let $H_0^p(F; G)$ be the subgroup of $H^p(F; G)$ of elements invariant under all the operations of $\pi_1(X)$. It is well known that

$$Z^0(X; H^p(F_\sigma; G_\sigma)) \approx H_0^p(F; G).$$

Of particular interest are those elements of $H_0^p(F; G)$ which correspond to elements of $Z^0(X; H^p(F_\sigma; G_\sigma))$ which can be lifted p -stages.

Let d be the smallest positive integer for which $H^d(F; G)$ is nontrivial. This means that for $0 < q < d$, $H^d(B_q, B_{q-1}; G_b) = 0$ and $H^{d+1}(B_{q+1}, B_q; G_b) = 0$.

It follows that in the main diagram the sequence

$$\begin{array}{ccc} H^d(B_d, B_{d-1}; G_b) & \xrightarrow{j^*} & H^d(B_d; G_b) \\ & \downarrow (i^*)^d & \\ H^d(B_0; G_b) & \xrightarrow{\delta^*} & H^{d+1}(B_1, B_0; G_b) \end{array}$$

is exact. Hence, any $u \in H_0^d(F; G)$ corresponds to an element of $H^d(B_0; G_b)$ that can be lifted d -stages. Moreover, if u can be lifted d -stages to $\bar{u}_1$ and $\bar{u}_2$ it is clear that $\bar{u}_1 - \bar{u}_2 \in \delta^* j^* H^d(B_d, B_{d-1}; G_b)$. Hence, $\lambda \bar{u}_1$ and $\lambda \bar{u}_2$ are cohomologous cocycles of $C^{d+1}(X; H^0(F_\sigma; G_\sigma))$, which is isomorphic to $C^{d+1}(X; G_\sigma)$. In this way we have a well defined homomorphism

$$T: H_0^d(F; G) \rightarrow H^{d+1}(X; G_\sigma).$$

As an application of this homomorphism let $\pi_p(F) = 0$ for $p < d$ (where $d > 1$) and assume $\pi_d(F) \neq 0$. Then $\pi_d(F) \approx H_d(F)$. (Where no coefficients are indicated, integers are to be understood.) Let $G_b = H_d(F_b)$. In $H^d(F; H_d(F))$ consider the class ω whose value on any cycle z is the homology class of z . This class ω is the first obstruction to contracting F to a point and is invariant under any homeomorphism of F . Therefore, $\omega \in H_0^d(F; H_d(F))$ and so corresponds to an $\bar{\omega} \in H^d(B_0; H_d(F_\sigma))$ which can be lifted d -stages. For this element $T(\omega) = \Omega \in H^{d+1}(X; H_d(F_\sigma))$ can be shown to be minus the *characteristic class* of the bundle, a result originally proved by G. Whitehead [15].

4. We now show how some of the multiplicative relations between the groups of the main diagram for various coefficient systems can be found. If $f: B \rightarrow X$ is a bundle with fiber F and group Γ , then f induces in an obvious way a bundle $f': X \times B \rightarrow X \times X$ also with fiber F and group Γ .

Let $\{G_b\}$ be a local system on B simple over every fiber and let $\{G'_x\}$ be a local system on X . Then $\{G'_x \otimes G_b\}$ is a local system on $X \times B$ which is simple over every fiber. Let u be an element of $H^d(B_q, B_{q-1}; G_b)$ which can be lifted k -stages to $\bar{u} \in H^{d+1}(B_{q+k+1}, B_{q+k}; G_b)$. If $v \in H^p(X^p, X^{p-1}; G'_x)$ is a cocycle, then $v \times u \in H^{p+d}((X \times B)_{p+q}, (X \times B)_{p+q-1}; G'_x \otimes G_b)$ can be lifted k -stages to $v \times \bar{u} \in H^{p+d+1}((X \times B)_{p+q+k+1}, (X \times B)_{p+q+k}; G'_x \otimes G_b)$.

The main diagram associated with a bundle is natural in the sense that it maps properly under a cellular bundle map as is shown below. Let $f_1: B_1 \rightarrow X_1$ and $f_2: B_2 \rightarrow X_2$ be fiberings with the same fiber F and group Γ . Let $\phi: B_1 \rightarrow B_2$ be a cellular bundle map; that is, ϕ induces a cellular map $\phi_0: X_1 \rightarrow X_2$ such that commutativity holds in the diagram

$$\begin{array}{ccc} B_1 & \xrightarrow{\phi} & B_2 \\ f_1 \downarrow & & \downarrow f_2 \\ X_1 & \xrightarrow{\phi_0} & X_2 \end{array}$$

If $\{G_{b_2}\}$ is a local system on B_2 simple over every fiber, $\phi^{-1}\{G_{b_2}\} = \{G_{b_1}\}$ is a local system on B_1 simple over every fiber. Since ϕ_0 is cellular, ϕ induces a homomorphism ϕ^* of the main diagram for B_2 with $\{G_{b_2}\}$ as coefficients into the main diagram of B_1 with $\{G_{b_1}\}$ as coefficients. This induced homomorphism ϕ^* clearly has the property that if u is an element of a group of the main diagram of B_2 that can be lifted k -stages to $\bar{u}$ then ϕ^*u can be lifted k -stages in B_1 to $\phi^*\bar{u}$.

5. Now we turn our attention to sphere bundles. Let $f: B \rightarrow X$ be a fibering with fiber S^d and arbitrary group Γ of homeomorphisms of S^d .

Let $\{G_b\}$ be a local system on B simple over every fiber. Since $H^p(F; G) = 0$ if $0 < p < d$ and $H^0(F; G) \approx H^d(F; G) \approx G$, we see that the main diagram reduces to the following:

$$\begin{array}{ccccccc}
 & & & & 0 & \xrightarrow{j^*} & H^p(B; G_b) \\
 & & & & & & \downarrow i^* \\
 & \cdots & \xrightarrow{\delta^*} & H^p(B_p, B_{p-1}; G_b) & \xrightarrow{j^*} & H^p(B_p; G_b) & \xrightarrow{\delta^*} H^{p+1}(B_{p+1}, B_p; G_b) \xrightarrow{j^*} \cdots \\
 & & & & & & \downarrow i^* \\
 & \cdots & \xrightarrow{\delta^*} & H^p(B_{p-d}, B_{p-d-1}; G_b) & \xrightarrow{j^*} & H^p(B_{p-d}; G_b) & \xrightarrow{\delta^*} H^{p+1}(B_{p-d+1}; B_{p-d}; G_b) \\
 & & & & & & \downarrow i^* \\
 & & & & & & 0
 \end{array}$$

where again any sequence composed of j^* , i^* , δ^* in that cyclic order is exact.

Let $G'_b = H_d(F_b) \otimes G_b$. Then $\{G'_b\}$ is a local system on B simple over every fiber, and $H_d(F_b) \otimes G'_b \approx G_b$. It is clear that $H^0(F_b; G_b) \approx G_b$ and that $H^d(F_b; G_b) \approx G'_b$.

From the above diagram we see that any element of $H^p(B_{p-d}, B_{p-d-1}; G_b)$ in the kernel of δ^*j^* can be lifted d -stages to an element of $H^{p+1}(B_{p+1}, B_p; G_b)$. This lifting induces a homomorphism

$$\Phi: H^{p-d}(X; G'_\sigma) \rightarrow H^{p+1}(X; G_\sigma).$$

Another homomorphism can be defined as follows. Let $u \in H^p(B; G_b)$. Then $\lambda j^{*-1}(i^*)^2(u)$ is a class of cohomologous cocycles in $C^{p-d}(X; G'_\sigma)$. This class in X depends only on u so that we have a homomorphism

$$\Psi: H^p(B; G_b) \rightarrow H^{p-d}(X; G'_\sigma).$$

The main theorem on sphere bundles is that the sequence

$$\cdots \xrightarrow{\Phi} H^p(X; G_\sigma) \xrightarrow{f^*} H^p(B; G_b) \xrightarrow{\Psi} H^{p-d}(X; G'_\sigma) \xrightarrow{\Phi} H^{p+1}(X; G_\sigma) \cdots$$

is exact. This shows that the cohomology groups of B are determined up to a group extension by the cohomology groups of X and the homomorphism Φ .

To calculate the mapping Φ let D be a cellular bundle map of B into $X \times B$ such that D_0 is a cellular approximation to the diagonal map of X into $X \times X$. If $v \in H^p(B_{p-d}, B_{p-d-1}; G_b)$ such that $\delta^* j^* v = 0$, then λv is a cocycle of $C^{p-d}(X; G'_\sigma)$ and $D^*(\lambda v \times \bar{\omega}) = v$ where $\bar{\omega} \in H^d(B_0; H_d(F_\sigma))$ corresponds to $\omega \in H^d_0(F; H_d(F))$, which was defined in §3. Since $\bar{\omega}$ can be lifted d -stages to $\bar{u}$ where $\lambda \bar{u}$ is in the class $\Omega = T(\omega)$, it follows that v can be lifted d -stages to $D^*(\lambda v \times \bar{u})$. Now $\lambda D^*(\lambda v \times \bar{u}) = \lambda v \cup \lambda \bar{u}$. Therefore, we have

$$\Phi(v) = v \cup \Omega \quad \text{for } v \in H^{p-d}(X; G'_\sigma).$$

If d is even, the same map D can be used to show that $2\Omega = 0$. These results contain those of Gysin [3] and are similar to some of the theorems of Thom [13] and Shapiro [9].

6. In this section we assume the coefficient system is simple and shall denote it by G . Let $l: F \rightarrow B$ be the injection of F into a fiber of B . We assume there is a homomorphism $\mu: H^p(F; G) \rightarrow H^p(B; G)$ such that $l^* \mu$ is the identity automorphism of $H^p(F; G)$ for all $p \geq 0$. If $H^p(F; G)$ is without torsion, this condition is equivalent to assuming l^* maps $H^p(B; G)$ onto $H^p(F; G)$, but, in general, it is stronger than the latter.

Let $H^p_0(B_q; G)$ be the kernel of δ^* in $H^p(B_q; G)$. We shall prove by induction on q the existence of homomorphisms

$$\mu_q: H^p_0(B_q; G) \rightarrow H^p(B_n; G)$$

such that:

- (a) $(i^*)^{n-p} \mu_q$ is the identity automorphism of $H^p_0(B_q; G)$.
- (b) $\mu_{q+1}(i^*)^{n-q-1} \mu_q = \mu_q$.

Since X is connected, $H^p_0(B_0; G) \approx H^p(F; G)$, and the existence of μ_0 with the required properties follows from the existence of μ . Assume μ_q has been constructed for $q < k$. To construct μ_k note that (b) defines μ_k on the direct summand $(i^*)^{n-k} \mu_{k-1} H^p_0(B_{k-1}; G)$ of $H^p_0(B_k; G)$. The other summand of $H^p_0(B_k; G)$ is isomorphic to $H^q(X; H^{p-q}(F; G))$. By using a cellular bundle map $D: B \rightarrow X \times B$ as in §5, the homomorphism μ_k can be defined on this latter summand in such a way that conditions (a) and (b) hold.

Having constructed the homomorphisms μ_q for all $0 \leq q \leq n$, it follows by induction on m that $H^p_0(B_m; G) \approx \sum_{q=0}^m H^q(X; H^{p-q}(F; G))$. Therefore, since $H^p(B; G) \approx H^p_0(B_n; G)$, we see that

$$H^p(B; G) \approx H^p(X \times F; G).$$

This result was announced by Hirsch [5] for the case where G is a field.

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UNIVERSITY OF CHICAGO,
CHICAGO, ILL., U. S. A.

DIFFERENTIABLE MANIFOLDS

DIFFERENTIAL GEOMETRY OF FIBER BUNDLES

SHIING-SHEN CHERN

The aim of this lecture is to give a discussion of the main results and ideas concerning a certain aspect of the so-called differential geometry in the large which has made some progress in recent years. Differential geometry in the large in its vaguest sense is concerned with relations between global and local properties of a differential-geometric object. In order that the methods of differential calculus may be applicable, the spaces under consideration are not only topological spaces but are differentiable manifolds. The existence of such a differentiable structure allows the introduction of notions as tangent vector, tangent space, differential forms, etc. In problems of differential geometry there is usually an additional structure such as: (1) a Riemann metric, that is, a positive definite symmetric covariant tensor field of the second order; (2) a system of paths with the property that through every point and tangent to every direction through the point there passes exactly one path of the system; (3) a system of cones of directions, one through each point, which correspond to the light cones in general relativity theory, etc. Among such so-called geometric objects the Riemann metric is perhaps the most important, both in view of its rôle in problems of analysis, mechanics, and geometry, and its richness in results. In 1917 Levi-Civita discovered his celebrated parallelism which is an infinitesimal transportation of tangent vectors preserving the scalar product and is the first example of a connection. The salient fact about the Levi-Civita parallelism is the result that it is the parallelism, and not the Riemann metric, which accounts for most of the properties concerning curvature.

The Levi-Civita parallelism can be regarded as an infinitesimal motion between two infinitely near tangent spaces of the Riemann manifold. It was Elie Cartan who recognized that this notion admits an important generalization, that the spaces for which the infinitesimal motion is defined need not be the tangent spaces of a Riemann manifold, and that the group which operates in the space plays a dominant rôle. In his theory of generalized spaces (*Espaces généralisés*) Cartan carried out in all essential aspects the local theory of what we shall call connections [1; 2]. With the development of the theory of fiber bundles in topology, begun by Whitney for the case of sphere bundles and developed by Ehresmann, Steenrod, Pontrjagin, and others, [8; 19], it is now possible to give a modern version of Cartan's theory of connections, as was first carried out by Ehresmann and Weil [7; 22].

Let F be a space acted on by a topological group G of homeomorphisms. A fiber bundle with the director space F and structural group G consists of topological spaces B , X and a mapping ψ of B onto X , together with the following:

- (1) X is covered by a family of neighborhoods $\{U_\alpha\}$, called the coordinate

neighborhoods, and to each U_α there is a homeomorphism (a coordinate function) $\varphi_\alpha: U_\alpha \times F \rightarrow \psi^{-1}(U_\alpha)$, with $\psi\varphi_\alpha(x, y) = x$, $x \in U_\alpha$, $y \in F$.

(2) As a consequence of (1), a point of $\psi^{-1}(U_\alpha)$ has the coordinates (x, y) , and a point of $\psi^{-1}(U_\alpha \cap U_\beta)$ has two sets of coordinates (x, y) and (x, y') , satisfying $\varphi_\alpha(x, y) = \varphi_\beta(x, y')$. It is required that $g_{\alpha\beta}(x): y' \rightarrow y$ be a continuous mapping of $U_\alpha \cap U_\beta$ into G .

The spaces X and B are called the base space and the bundle respectively. Each subset $\psi^{-1}(x) \subset B$ is called a fiber.

This definition of a fiber bundle is too narrow in the sense that the coordinate neighborhoods and coordinate functions form a part of the definition. An equivalence relation has thus to be introduced. Two bundles (B, X) , (B', X) with the same base space X and the same F, G are called equivalent if, $\{U_\alpha, \varphi_\alpha\}$, $\{V_\beta, \theta_\beta\}$ being respectively their coordinate neighborhoods and coordinate functions, there is a fiber-preserving homeomorphism $T: B \rightarrow B'$ such that the mapping $h_{\alpha\beta}(x): y \rightarrow y'$ defined by $\theta_\beta(x, y) = T\varphi_\alpha(x, y')$ is a continuous mapping of $U_\alpha \cap V_\beta$ into G .

An important operation on fiber bundles is the construction from a given bundle of other bundles with the same structural group, in particular, the principal fiber bundle which has G as director space acted upon by G itself as the group of left translations. The notion of the principal fiber bundle has been at the core of Cartan's method of moving frames, although its modern version was first introduced by Ehresmann. It can be defined as follows: For $x \in X$, let G_x be the totality of all maps $\varphi_{\alpha,g}(x): F \rightarrow \psi^{-1}(x)$ defined by $y \rightarrow \varphi_\alpha(x, g(y))$, $y \in F$, $g \in G$, relative to a coordinate neighborhood U_α containing x . G_x depends only on x . Let $B^* = \bigcup_{x \in X} G_x$ and define the mapping $\psi^*: B^* \rightarrow X$ by $\psi^*(G_x) = x$ and the coordinate functions $\varphi_\alpha^*(x, g) = \varphi_{\alpha,g}(x)$. Topologize B^* such that the φ_α^* 's define homeomorphisms of $U_\alpha \times G$ into B^* . The bundle (B^*, X) so obtained is called a principal fiber bundle. This construction is an operation on the equivalence classes of bundles in the sense that two fiber bundles are equivalent if and only if their principal fiber bundles are equivalent. Similarly, an inverse operation can be defined, which will permit us to construct bundles with a given principal bundle and having as director space a given space acted upon by the structural group G . An important property of the principal fiber bundle is that B^* is acted upon by G as right translations.

For the purpose of differential geometry we shall assume that all spaces under consideration are differentiable manifolds and that our mappings are differentiable with Jacobian matrices of the highest rank everywhere. In particular, the structural group G will be assumed to be a connected Lie group. For simplicity we suppose our base space X to be compact, although a large part of our discussions holds without this assumption.

The implications of these assumptions are very far-reaching indeed. First of all we can draw into consideration the Lie algebra $L(G)$ of G . $L(G)$ is invariant under the left translations of G , while the right translations and the inner automorphisms of G induce on $L(G)$ a group of linear endomorphisms $\text{ad}(G)$, called

the adjoint group of G . Relative to a base of $L(G)$ there are the left-invariant linear differential forms ω^i and the right-invariant linear differential forms π^i , each set consisting of linearly independent forms whose number is equal to the dimension of G . A fundamental theorem on Lie groups asserts that their exterior derivatives are given by

$$(1) \quad \begin{aligned} d\omega^i &= -\frac{1}{2} \sum_{j,k} c_{jk}^i \omega^j \wedge \omega^k, \\ d\pi^i &= +\frac{1}{2} \sum_{j,k} c_{jk}^i \pi^j \wedge \pi^k, \quad i, j, k = 1, \dots, \dim G, \end{aligned}$$

where c_{jk}^i are the so-called constants of structure which are antisymmetric in the lower indices and which satisfy the well-known Jacobi relations.

Returning to our fiber bundle, the dual mapping of the mapping $g_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow G$ carries ω^i and π^i into linear differential forms in $U_\alpha \cap U_\beta$, which we shall denote by $\omega_{\alpha\beta}^i$ and $\pi_{\alpha\beta}^i$ respectively. Since $g_{\alpha\gamma} = g_{\alpha\beta}g_{\beta\gamma}$ in $U_\alpha \cap U_\beta \cap U_\gamma$, we have

$$(2) \quad \omega_{\alpha\gamma}^i = \sum_j \text{ad}(g_{\beta\gamma})_j^i \omega_{\alpha\beta}^j + \omega_{\beta\gamma}^i.$$

We can also interpret $\omega_{\alpha\beta}^i$ as a vector-valued linear differential form in $U_\alpha \cap U_\beta$, with values in $L(G)$, and shall denote it simply by $\omega_{\alpha\beta}$ when so interpreted.

The generalization of the notion of a tensor field in classical differential geometry leads to the following situation: Let E be a vector space acted on by a representation $M(G)$ of G . A tensorial differential form of degree r and type $M(G)$ is an exterior differential form u_α of degree r in each coordinate neighborhood U_α , with values in E , such that, in $U_\alpha \cap U_\beta$, $u_\alpha = M(g_{\alpha\beta})u_\beta$. The exterior derivative du_α of u_α is in general not a tensorial differential form. It is in order to preserve the tensorial character of the derivative that an additional structure, a connection, is introduced into the fiber bundle.

A connection in the fiber bundle is a set of linear differential forms θ_α in U_α , with values in $L(G)$, such that

$$(3) \quad \omega_{\alpha\beta} = -\text{ad}(g_{\alpha\beta})\theta_\alpha + \theta_\beta, \quad \text{in } U_\alpha \cap U_\beta.$$

It follows from (2) that such relations are consistent in $U_\alpha \cap U_\beta \cap U_\gamma$. As can be verified without difficulty, a connection defines in the principal fiber bundle a field of tangent subspaces transversal to the fibers, that is, tangent subspaces which, together with the tangent space of the fiber, span at every point the tangent space of the principal bundle. It follows from elementary extension theorems that in every fiber bundle there can be defined a connection. As there is great freedom in the choice of the connection, the question of deciding the relationship between the properties of the bundle and those of the connection will be our main concern in this paper.

Let us first define the process of so-called absolute differentiation. Let $\bar{M}(X)$, $X \in L(G)$, be the representation of the Lie algebra $L(G)$ induced by the repre-

sentation $M(G)$ of G . Then we have

$$(4) \quad dM(g_{\alpha\beta}) = M(g_{\alpha\beta})\bar{M}(\theta_\beta) - \bar{M}(\theta_\alpha)M(g_{\alpha\beta}).$$

It follows that if we put for our tensorial differential form u_α of degree r and type $M(G)$

$$(5) \quad Du_\alpha = du_\alpha + \bar{M}(\theta_\alpha) \wedge u_\alpha,$$

the form Du_α will be a tensorial differential form of degree $r + 1$ and the same type $M(G)$.

To study the local properties of the connection we again make use of a base of the Lie algebra, relative to which the form θ_α has the components θ_α^i . We put

$$(6) \quad \Theta_\alpha^i = d\theta_\alpha^i + \frac{1}{2} \sum_{j,k} c_{jk}^i \theta_\alpha^j \wedge \theta_\alpha^k, \quad \text{in } U_\alpha.$$

The form Θ_α , whose components relative to the base are Θ_α^i , is then an exterior quadratic differential form of degree 2, with values in $L(G)$. It is easy to verify that $\Theta_\alpha = \text{ad}(g_{\alpha\beta})\Theta_\beta$ in $U_\alpha \cap U_\beta$. The Θ_α 's therefore define a tensorial differential form of degree 2 and type $\text{ad}(G)$, called the curvature tensor of the connection. In a manner which we shall not attempt to describe here, the curvature tensor and tensors obtained from it by successive absolute differentiations give all the local properties of the connection. In particular, the condition $\Theta_\alpha = 0$ is a necessary and sufficient condition for the connection to be flat, that is, to be such that $\theta_\alpha = 0$ by a proper choice of the coordinate functions.

The following formulas for absolute differentiation can easily be verified:

$$(7) \quad \begin{aligned} \bar{M}(\Theta_\alpha) &= d\bar{M}(\theta_\alpha) + \bar{M}(\theta_\alpha)^2, \\ D\Theta_\alpha &= 0, \\ D^2u_\alpha &= \bar{M}(\Theta_\alpha)u_\alpha. \end{aligned}$$

Such relations are known in classical cases, the second as the Bianchi identity.

We now consider real-valued symmetric multilinear functions $P(Y_1, \dots, Y_k)$, $Y_i \in L(G)$, $i = 1, \dots, k$, which are invariant, that is, which are such that $P(\text{ad}(a)Y_1, \dots, \text{ad}(a)Y_k) = P(Y_1, \dots, Y_k)$ for all $a \in G$. For simplicity we shall call such a function an invariant polynomial, k being its degree. By the definition of addition,

$$(8) \quad (P + Q)(Y_1, \dots, Y_k) = P(Y_1, \dots, Y_k) + Q(Y_1, \dots, Y_k),$$

all invariant polynomials of degree k form an abelian group. Let $I(G)$ be the direct sum of these abelian groups for all $k \geq 0$. If P and Q are invariant polynomials of degrees k and l respectively, we define their product PQ to be an invariant polynomial of degree $k + l$ given by

$$(9) \quad (PQ)(Y_1, \dots, Y_{k+l}) = \frac{1}{N} \sum P(Y_{i_1}, \dots, Y_{i_k})Q(Y_{i_{k+1}}, \dots, Y_{i_{k+l}}),$$

where the summation is extended over all permutations of the vectors Y_i , and N is the number of such permutations. This definition of multiplication, together with the distributive law, makes $I(G)$ into a commutative ring, the ring of invariant polynomials of G .

Let $P \in I(G)$, with degree k . For Y_i we substitute the curvature tensor Θ . Then $P(\Theta) = P(\Theta, \dots, \Theta)$ is an exterior differential form of degree $2k$, which, because of the invariance property of P , is defined everywhere in the base space X . From the Bianchi identity (7₂) it follows that $P(\Theta)$ is closed. Therefore, by the de Rham theory, $P(\Theta)$ determines an element of the cohomology ring $H(X)$ of X having as coefficient ring the field of real numbers. This mapping is a ring homomorphism

$$(10) \quad h: I(G) \rightarrow H(X)$$

of the ring of invariant polynomials of G into the cohomology ring of X . It is defined with the help of a connection in the bundle.

Our first main result is the following theorem of Weil: h is independent of the choice of the connection [22]. In other words, two different connections in the fiber bundle give rise to the same homomorphism h . To prove this we notice that if θ_α and θ'_α are the linear differential forms defining these connections, their difference $u_\alpha = \theta'_\alpha - \theta_\alpha$ is a linear differential form of type $\text{ad}(G)$, with values in $L(G)$. With the help of u_α Weil constructs a differential form whose exterior derivative is equal to the difference $P(\Theta') - P(\Theta)$, for a given invariant polynomial P . Another proof has been given recently by H. Cartan, by means of an invariant definition of the homomorphism h .

Our next step consists in setting up a relationship between this homomorphism h and a homomorphism which is defined in a purely topological manner. This requires the concepts of an induced fiber bundle and a universal fiber bundle.

Let a mapping $f: Y \rightarrow X$ be given. The neighborhoods $\{f^{-1}(U_\alpha)\}$ then form a covering of Y and coordinate functions $\varphi'_\alpha: f^{-1}(U_\alpha) \times F \rightarrow f^{-1}(U_\alpha) \times \psi^{-1}(U_\alpha)$ can be defined by $\varphi'_\alpha(\eta, y) = \eta \times \varphi_\alpha(f(\eta), y)$. This defines a fiber bundle $Y \times \psi^{-1}(f(Y))$ over Y , with the same director space F and the same group G . The new bundle is said to be induced by the mapping f . If the original bundle has a connection given by the differential form θ_α in U_α , the dual mapping f^* of f carries θ_α into $f^*\theta_\alpha$ in $f^{-1}(U_\alpha)$ for which the relation corresponding to (3) is valid. The forms $f^*\theta_\alpha$ therefore define an induced connection in the induced bundle.

This method of generating new fiber bundles from a given bundle is very useful. Its value is based on the fact that it provides a way for the enumeration of fiber bundles. In fact, let the director space and the structural group G be given and fixed for our present considerations. A bundle with the base space X_0 is called universal relative to a space X if every bundle over X is equivalent to a bundle induced by a mapping $X \rightarrow X_0$ and if two such induced bundles are equivalent when and only when the mappings are homotopic. If, for a space X , there exists a universal bundle with the base space X_0 , then the classes of bundles over X are in one-one correspondence with the homotopy classes of mappings $X \rightarrow X_0$,

so that the enumeration of the bundles over X reduces to a homotopy classification problem.

It is therefore of interest to know the circumstances under which a universal bundle exists. A sufficient condition for the bundle over X_0 to be universal for all compact spaces X of dimension less than or equal to n is that the bundle B_0 of its principal fiber bundle have vanishing homotopy groups up to dimension n inclusive: $\pi_i(B_0) = 0$, $0 \leq i \leq n$, where the condition $\pi_0 = 0$ means connectedness.

Under our assumptions that X is compact and that G is a connected Lie group, bundles can be found such that these conditions are fulfilled. First of all, according to a theorem due to E. Cartan, Malcev, Iwasawa, and Mostow, [12; 14; 15], G contains a maximal compact subgroup G_1 , and the homogeneous space G/G_1 is homeomorphic to a Euclidean space. This makes it possible to reduce problems of equivalence, classification, etc. of bundles with the group G to the corresponding problems for G_1 . Since G_1 is a compact Lie group, it has a faithful orthogonal representation and can be considered as a subgroup of the rotation group $R(m)$ operating in an m -dimensional Euclidean space E^m . Imbed E^m in an $(m + n + 1)$ -dimensional Euclidean space E^{m+n+1} and consider the homogeneous space $\tilde{B} = R(m + n + 1)/(I_m \times R(n + 1))$ as a bundle over $X_0 = R(m + n + 1)/(G_1 \times R(n + 1))$, where I_m is the identical automorphism of E^m , and $R(n + 1)$ is the rotation group of the space E^{n+1} perpendicular to E^m in E^{m+n+1} . This is a principal bundle with G_1 as its structural group. By the covering homotopy theorem we can prove that $\pi_i(\tilde{B}) = 0$, $0 \leq i \leq n$. In this way the existence of a universal bundle is proved by an explicit construction.

Suppose that a universal bundle exists, with the base space X_0 . Let $H(X, R)$ be the cohomology ring of X , relative to the coefficient ring R . Since the classes of bundles over X are in one-one correspondence with the homotopy classes of mappings $X \rightarrow X_0$, the homomorphism $h': H(X_0, R) \rightarrow H(X, R)$ is completely determined by the bundle. h' will be called the characteristic homomorphism, its image $h'(H(X_0, R)) \subset H(X, R)$ the characteristic ring, and an element of the characteristic ring a characteristic cohomology class. It will be understood that the coefficient ring R will be the field of real numbers whenever it is dropped in the notation.

The universal bundle is of course not unique. However, given any two bundles which are universal for compact base-spaces of dimension less than or equal to n , it is possible to establish between their base spaces X_0 and X'_0 a chain transformation of the singular chains of dimension less than or equal to n which gives rise to a chain equivalence. From this it follows that up to the dimension n inclusive, the cohomology rings of X_0 and X'_0 are in a natural isomorphism. The characteristic homomorphism is therefore independent of the choice of the universal bundle. Although this conclusion serves our purpose, it may be remarked that, in terms of homotopy theory, a stronger result holds between X_0 and X'_0 , namely, they have the same homotopy- n -type. From this the above assertion follows as a consequence.

A knowledge of $H(X_0, R)$ would be necessary for the description of the characteristic homomorphism. Since elements of dimension greater than n ($= \dim X$) of $H(X_0, R)$ are mapped into zero by dimensional considerations, $H(X_0, R)$ can be replaced by any ring which is isomorphic to it up to dimension n inclusive. On the other hand, it follows from the discussions of the last section that the choice of the universal bundle is immaterial, so that we can take the one whose base space is $X_0 = R(m + n + 1)/(G_1 \times R(n + 1))$. Using a connection in this universal bundle, we can, according to a process given above, define a homomorphism $h_0 = I(G_1) \rightarrow H(X_0)$ of the ring of invariant polynomials of G_1 into $H(X_0)$. X_0 being a homogeneous space, its cohomology ring $H(X_0)$ with real coefficients can be studied algebraically by methods initiated by E. Cartan and recently developed with success by H. Cartan, Chevalley, Kozsul, Leray, and Weil [13]. Thus it has been shown that, up to dimension n , h_0 is a one-one isomorphism. We may therefore replace $H(X_0)$ by $I(G_1)$ in the homomorphism h' and write the characteristic homomorphism as

$$(11) \quad h': I(G_1) \rightarrow H(X).$$

This homomorphism h' is defined by the topological properties of the fiber bundle.

On the other hand, the homomorphism $h: I(G) \rightarrow H(X)$ defined above can be split into a product of two homomorphisms. Since an invariant polynomial under G is an invariant polynomial under G_1 , there is a natural homomorphism

$$(12) \quad \sigma: I(G) \rightarrow I(G_1).$$

Since G_1 can be taken to be the structural group, the homomorphism

$$(13) \quad h_1: I(G_1) \rightarrow H(X)$$

is defined. Now, a connection with the group G_1 can be considered as a connection with the group G . Using such a connection, we can easily prove

$$(14) \quad h = h_1 \sigma.$$

Our main result which seems to include practically all our present knowledge on the subject consists in the statement:

$$(15) \quad h' = h_1.$$

Notice that h' is defined by the topological properties of the bundle and h_1 by the help of a connection, so that our theorem gives a relationship between a bundle and a connection defined in it, which is restrictive in one way or the other. In particular, when the structural group G is compact, we have $G_1 = G$ and σ is the identity, and the characteristic homomorphism is in a sense determined by the connection. For instance, it follows that the characteristic ring of the bundle has to be zero when a connection can be defined such that $h(I(G)) = 0$.

A proof for this theorem is obtained by first establishing it for the universal bundle. Under the mapping $f: X \rightarrow X_0$ it is then true for the induced bundle and the induced connection. Using the theorem of Weil that h is independent of the choice of the connection, we see that the relation is true for any connection in the bundle.

A great deal can be said about the rings of invariant polynomials $I(G)$, $I(G_1)$ and the homomorphism σ . When the structural group is compact, such statements can usually be proved more simply by topological considerations. In the other case we have to make use of the cohomology theory of Lie algebras. As we do not wish to discuss this, we shall restrict ourselves to the explanation of the corresponding topological notions. For this purpose we shall first discuss compact groups, that is, we begin by confining our attention to G_1 .

We first recall some results on compact group manifolds. All the maximal abelian subgroups are conjugate and are isomorphic to a torus whose dimension is called the rank of the group. By an idea due essentially to Pontrjagin [16] we can define an operation of the homology classes of G_1 on the cohomology classes of G_1 . In fact, $m: G_1 \times G_1 \rightarrow G_1$ being defined by the group multiplication, the image $m^*\gamma^k$ of a cohomology class of dimension k of G under the dual homomorphism m^* can be written $m^*\gamma^k = \sum u_i^r \times v_i^{k-r}$. The operation of a homology class c^s of dimension $s \leq k$ on γ^k is then defined as $i(c^s)\gamma^k = \sum_i KI(c^s, u_i^s)v_i^{k-s}$. We call this operation an interior product. A cohomology class γ^k of G_1 is called primitive if its interior product by any homology class of dimension s , $1 \leq s \leq k-1$, is zero. The homology structure of compact group manifolds (with real coefficients) has a description given by the following theorem of Hopf and Samelson [11; 18]: (1) all primitive cohomology classes are of odd dimension; (2) the vector space of the primitive classes has as dimension the rank of G ; (3) the cohomology ring of G_1 is isomorphic to the Grassmann algebra of the space of primitive classes.

The primitive classes play a rôle in the study of the universal principal fiber bundle $\psi: B_0 \rightarrow X_0$. Identify a fiber $\psi^{-1}(x)$ ($x \in X_0$) with G_1 , and let i be the inclusion mapping of G_1 into B_0 . If γ^k is a cocycle of X_0 , $\psi^*\gamma^k$ is a cocycle of B_0 . Since B_0 is homologically trivial, there exists a cochain β^{k-1} having $\psi^*\gamma^k$ as co-boundary. Then $i^*\beta^{k-1}$ is a cocycle in G_1 whose cohomology class depends only on that of γ^k . The resulting mapping of the cohomology classes is called a transgression. It is an additive homomorphism of the ring of invariant polynomials of G_1 into the cohomology ring of G_1 and it carries an invariant polynomial of degree k into a cohomology class of dimension $2k-1$. Chevalley and Weil proved that the image is precisely the space of the primitive classes.

When the group G is noncompact, the consideration of its Lie algebra allows us to generalize the above notions, at least under the assumption that G is semi-simple. H. Cartan, Chevalley, and Koszul have developed a very comprehensive theory dealing with the situation, which can be considered in a sense as the algebraic counterpart of the above treatment. Among their consequences we mention the following which is interesting for our present purpose: The ring of in-

variant polynomials under G has a set of generators equal to the rank of G ; these can be so chosen that their images under transgression span the space of primitive classes of G .

Using the fact that the cohomology theory of Lie algebras and transgression can be defined algebraically, and therefore for G , we have the following diagram

$$\begin{array}{ccc} H(G) & \xrightarrow{i^*} & H(G_1) \\ t \uparrow & & \uparrow t_1 \\ I(G) & \xrightarrow{\sigma} & I(G_1). \end{array}$$

It is not difficult to prove that commutativity holds in this diagram. Hence the image under σ depends on the image under i^* of $H(G)$, that is, on the "homological position" of G_1 in G . In general, $\sigma[I(G)] \neq I(G_1)$.

There are relations between the characteristic cohomology classes in our definition and the classes carrying the same name in the topological method of obstructions but we cannot discuss them in detail. The latter come into being when one attempts to define a cross-section in the fiber bundle (that is, a mapping f of X into B , such that ψf is the identity) by extension over the successive skeletons; they are cohomology classes over groups of coefficients which are the homotopy groups of the director space. As we shall see from examples, it is sometimes possible to identify them by identifying the coefficient groups. In general, however, our characteristic classes are based on homological considerations, while those of obstruction theory are based on homotopy considerations. Their rôles are complementary.

We shall devote the rest of this lecture to the consideration of examples. Although the main results will follow from the general theorems, special problems arise in individual cases which can be of considerable interest. To begin with, take for G the rotation group in m variables, and suppose that a connection is given in the bundle. This includes in particular the case of orientable Riemann manifolds with a positive definite metric, the bundle being the tangent bundle of the manifold and the connection being given by the parallelism of Levi-Civita; it also includes, among other things, the theory of orientable submanifolds imbedded in an orientable Riemann manifold.

By a proper choice of a base of the Lie algebra of $G = R(m)$, the space of the Lie algebra can be identified with the space of skew-symmetric matrices of order m . The connection can therefore be defined, in every coordinate neighborhood, by a skew-symmetric matrix of linear differential forms $\theta = (\theta_{ij})$, and its curvature tensor by a skew-symmetric matrix of quadratic differential forms $\Theta = (\Theta_{ij})$. The effect of the adjoint group is given by $\text{ad}(a)\Theta = A\Theta {}^tA$, where A is a proper orthogonal matrix and ${}^tA$ is its transpose.

The first question is of course to determine a set of generators for the ring of invariant polynomials; using the fundamental theorem on invariants, it is easy to do this explicitly [23]. Instead of the invariant polynomials we write the

corresponding differential forms:

$$\begin{aligned}
 \Delta_s &= \Theta_{i_1 i_2} \cdots \Theta_{i_s i_1}, & s &= 2, 4, \cdots, m+1, & m \text{ odd} \\
 \Delta_s &= \Theta_{i_1 i_2} \cdots \Theta_{i_s i_1}, & s &= 2, 4, \cdots, m-2, & m \text{ even} \\
 \Delta_0 &= \epsilon_{i_1 \cdots i_m} \Theta_{i_1 i_2} \cdots \Theta_{i_{m-1} i_m}, & & & m \text{ even,}
 \end{aligned}
 \tag{16}$$

where repeated indices imply summation and where $\epsilon_{i_1 \cdots i_m}$ is the Kronecker tensor, equal to $+1$ or -1 according as $i_1, \cdots, i_m$ form an even or odd permutation of $1, \cdots, m$ and otherwise to 0 . Since the rank of $R(m)$ is $(m+1)/2$ or $m/2$ according as m is odd or even, we verify here that the number of the above generators is equal to the rank. They form a complete set of generators, because they are obviously independent.

It follows that the cohomology classes determined by these differential forms or by polynomials in these differential forms depend only on the bundle and not on the connection. As a consequence, if all these differential forms are zero, the characteristic ring is trivial. The differential forms in (16) were first given by Pontrjagin [17].

For geometric applications it is useful to have a more explicit description of the base space of a universal bundle. This is all the more significant, since it would then allow us to study the characteristic homomorphisms with coefficient rings other than the field of real numbers. Our general theory gives as such a base space the Grassmann manifold

$$X_0 = R(m+n+1)/(R(m) \times R(n+1)),$$

which can be identified with the space of all oriented m -dimensional linear spaces through a point 0 of an $(m+n+1)$ -dimensional Euclidean space E^{m+n+1} .

The homology structure of Grassmann manifolds has been studied by Ehresmann [9, 10]. A cellular decomposition can be constructed by the following process: Take a sequence of linear spaces

$$0 \subset E^1 \subset E^2 \subset \cdots \subset E^{m+n} \subset E^{m+n+1}.$$

Corresponding to a set of integers

$$0 \leq a_1 \leq a_2 \leq \cdots \leq a_m \leq n+1,$$

denote by $(a_1 \cdots a_m)$ the set of all m -dimensional linear spaces $\xi \in X_0$ such that

$$\dim (\xi \cap E^{a_i+i}) \geq i, \quad i = 1, \cdots, m.$$

The interior points of $(a_1 \cdots a_m)$ form two open cells of dimension $a_1 + \cdots + a_m$. These open cells constitute a cellular decomposition of X_0 , whose incidence relations can be determined. From this we can determine the homology and cohomology groups of X_0 . In particular, it follows that the symbol $(a_1 \cdots a_m)^\pm$ can be used to denote a cochain, namely, the one which has the value $+1$ for the corresponding open cells and has otherwise the value zero. The characteristic homomorphism can then be described as a homomorphism of

combinations of such symbols into the cohomology ring $H(X, R)$ of X . When R is the field of real numbers, the result is particularly simple. In fact, a base for the cohomology groups of dimensions less than or equal to n consists of cocycles having as symbols those for which all a_i are even, together with the cocycle $(1 \cdots 1)$ when m is even.

This new description of the characteristic homomorphism allows us to give a geometric meaning to individual characteristic classes. In this respect the class $h'((1 \cdots 1))$, which exists only when m is even, deserves special attention. In fact, the bundle with the director space $S^{m-1} = R(m)/R(m-1)$ constructed from the principal bundle is a bundle of $(m-1)$ -spheres in the sense of Whitney. For such a sphere bundle, Whitney introduced a characteristic cohomology class W^m with integer coefficients. It can be proved that W^m , when reduced to real coefficients, is precisely the class $h'((1 \cdots 1))$. On the other hand, the latter can be identified on the universal bundle with a numerical multiple of the class defined by the differential form Δ_0 . Taking the values of these classes for the fundamental cycle of the base manifold, we can write the result in an integral formula

$$(17) \quad W^m \cdot X = c \int_X \Delta_0,$$

where c is a numerical factor and X denotes a fundamental cycle of the base manifold. For a Riemann manifold, $W^m \cdot X$ is equal to the Euler-Poincaré characteristic of X and our formula reduces to the Gauss-Bonnet formula [3].

We introduce the notations

$$(18) \quad \begin{aligned} P^{4k} &= h'(0 \cdots 0 \underbrace{2 \cdots 2}_{2k \text{ times}}) \\ \bar{P}^{4k} &= h'(0 \cdots 0 \ 2k \ 2k) \\ \chi^m &= h'(1 \cdots 1), \quad m \text{ even,} \end{aligned}$$

where the symbols denote also the cohomology classes to which the respective cocycles belong. By studying the multiplicative structure of the cohomology ring of X_0 , we can prove that the characteristic homomorphism is determined by the classes $P^{4k}, \chi^m, 4k \leq \dim X$ or the classes $\bar{P}^{4k}, \chi^m, 4k \leq \dim X$.

We shall mention an application of the classes $\bar{P}^{4k}$. Restricting ourselves for simplicity to the tangent bundle of a compact differentiable manifold, the conditions $\bar{P}^{4k} = 0, 2k \geq n+2$, are necessary for the manifold to be imbeddable into a Euclidean space of dimension $m+n+1$. We get thus criteria on the impossibility of imbedding which can be expressed in terms of the curvature tensor of a Riemann metric on the manifold.

The second example we shall take up is the case that G is the unitary group. Such bundles occur as tangent bundles of complex analytic manifolds, and the introduction of an Hermitian metric in the manifold would give rise to a connection in the bundle.

The space of the Lie algebra of the unitary group $U(m)$ in m variables can be identified with the space of $m \times m$ Hermitian matrices A (${}^t\bar{A} = A$). A connection is therefore defined in each coordinate neighborhood by an Hermitian matrix of linear differential forms $\theta = (\theta_{ij})$ and its curvature tensor by an Hermitian matrix of quadratic differential forms $\Theta = (\Theta_{ij})$. Under the adjoint group the curvature tensor is transformed according to $\text{ad}(a)\Theta = A\Theta{}^t\bar{A}$, A being a unitary matrix. Using this representation of the adjoint group, a set of invariant polynomials can be easily exhibited. We give their corresponding differential forms as

$$(19) \quad \Lambda_k = \Theta_{i_1 i_2} \cdots \Theta_{i_k i_1}, \quad k = 1, \cdots, m.$$

Since they are clearly independent and their number is equal to the rank m of $U(m)$, they form a complete set of generators in the ring of invariant polynomials.

As in the case of the rotation group the complex Grassmann manifold $X_0 = U(m+n)/(U(m) \times U(n))$ is the base space of a universal bundle, whose study would be useful for some geometric problems. The results are simpler than the real case, but we shall not describe them here. A distinctive feature of the complex case is that a set of generators can be chosen in the ring of invariant polynomials whose corresponding differential forms are

$$(20) \quad \Psi_r = \frac{1}{(2\pi (-1)^{1/2})^{m-r+1} (m-r+1)!} \sum \delta(i_1 \cdots i_{m-r+1}; j_1 \cdots j_{m-r+1}) \\ \cdot \Theta_{i_1 j_1} \cdots \Theta_{i_{m-r+1} j_{m-r+1}}, \quad r = 1, \cdots, m,$$

where $\delta(i_1 \cdots i_{m-r+1}; j_1 \cdots j_{m-r+1})$ is zero except when $j_1, \cdots, j_{m-r+1}$ form a permutation of $i_1, \cdots, i_{m-r+1}$, in which case it is $+1$ or -1 according as the permutation is even or odd, and where the summation is extended over all indices $i_1, \cdots, i_{m-r+1}$ from 1 to m . This set of generators has the advantage that the cohomology classes determined by the differential forms have a simple geometrical meaning. In fact, they are the classes, analogous to the Stiefel-Whitney classes, for the bundle with the director space $U(m)/U(m-r)$. As such they are primary obstructions to the definition of a cross-section and are therefore more easily dealt with [4]. Substantially the same classes have been introduced by M. Eger and J. A. Todd in algebraic geometry, even before they first made their appearance in differential geometry [6; 20].

The situation is different for bundles with the rotation group, since the Stiefel-Whitney classes, except the highest-dimensional one, are essentially classes mod 2 and therefore do not enter into our picture. However, there is a close relationship between bundles with the group $R(m)$ and bundles with the group $U(m)$. In fact, given a bundle with the group $R(m)$, we can take its Whitney product with itself, which is a bundle with the same base space and the group $R(m) \times R(m)$. The latter can be imbedded into $U(m)$, so that we get a bundle with the group $U(m)$. Such a process is frequently useful in reducing problems on bundles with the rotation group to those on bundles with the unitary group.

We shall take as last example the case that the group is the component of the identity of the general linear group $GL(m)$ in m variables. A connection in the bundle is called an affine connection. An essential difference from the two previous examples is that the group is here noncompact.

The Lie algebra of the group $GL(m)$ can be identified with the space of all m -rowed square matrices, so that the curvature tensor in each coordinate neighborhood is given by such a matrix of exterior quadratic differential forms: $\Theta = (\Theta_i^j)$. The effect of the adjoint group being defined by $\text{ad}(a)\Theta = A \Theta A^{-1}$, $a \in GL(m)$, it is easily seen that a set of generators of the ring of invariant polynomials can be so chosen that the corresponding differential forms are

$$(21) \quad M_s = \Theta_{i_1}^{i_2} \cdots \Theta_{i_s}^{i_1}, \quad s = 1, \dots, m-1.$$

According to the general theory it remains to determine the homomorphism of the ring of invariant polynomials under $GL(m)$ into the ring of invariant polynomials under its maximal compact subgroup, which is in this case the rotation group $R(m)$. It is seen that M_s , for even s , is mapped into Δ_s , and, for odd s , is mapped into zero. The class defined by Δ_0 does not belong to the image of the homomorphism. This fact leads to the interesting explanation that a formula analogous to the Gauss-Bonnet formula does not exist for an affine connection.

Perhaps the most important of the bundles is the tangent bundle of a differentiable manifold. We mentioned above the identification of a certain characteristic class with the Euler-Poincaré characteristic of the manifold, at least for the case that the manifold is orientable and of even dimension. Beyond this very little is known on the relations between topological invariants of the manifold and the characteristic homomorphism of its tangent bundle. Recently, contributions have been made by Thom and Wu which bear on this question [21; 25]. Although it is not known whether a topological manifold always has a differentiable structure, nor whether it can have two essentially different differentiable structures, Thom and Wu proved that the characteristic homomorphisms of the tangent bundle, with coefficients mod 2 and with coefficients mod 3, are independent of the choice of the differentiable structure, provided one exists. Briefly speaking, this means that such characteristic homomorphisms are topological invariants of differentiable manifolds. The proof for coefficients mod 3 is considerably more difficult than the case mod 2.

For bundles with other groups such questions have scarcely been asked. The next case of interest is perhaps the theory of projective connections derived from the geometry of paths. In this case the bundle with the projective group depends both on the tangent bundle and the family of paths. It would be of interest to know whether or what part of the characteristic homomorphism is a topological invariant of the manifold.

Before concluding we shall mention a concept which has no close relation with the above discussion, but which should be of importance in the theory of connections, namely, the notion of the group of holonomy. It can be defined as follows: if ω is the left-invariant differential form in G , with values in $L(G)$,

and if θ_α defines a connection, the equation

$$(22) \quad \theta_\alpha + \omega = 0$$

is independent of the coordinate neighborhood. When a parametrized curve is given in the base manifold, this differential equation defines a family of integral curves in G invariant under left translations of the group. Let $x \in X$ and consider all closed parametrized curves in X having x as the initial point. To every such curve C let $a(C)$ be the endpoint of the integral curve which begins at the unit element e of G . All such points $a(C)$ form a subgroup H of G , the group of holonomy of the connection.

Added in proof: The details of some of the discussions in this article can be found in mimeographed notes of the author, *Topics in differential geometry*, Institute for Advanced Study, Princeton, 1951.

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UNIVERSITY OF CHICAGO,
CHICAGO, ILL., U. S. A.

SUR LES VARIÉTÉS PRESQUE COMPLEXES

CHARLES EHRESMANN

1. Introduction. Etant donnée une variété topologique V_{2n} , de dimension $2n$, existe-t-il sur V_{2n} une *structure analytique complexe*? Plus abordable paraît la question suivante: Etant donnée une variété *différentiable* V_{2n} , existe-t-il sur V_{2n} une structure analytique complexe *subordonnée à sa structure différentiable*? Soit $T(V_{2n})$ l'espace des vecteurs tangents à V_{2n} et T_x l'espace vectoriel tangent en $x \in V_{2n}$. $T(V_{2n})$ est un espace fibré de base V_{2n} et de fibres T_x isomorphes à l'espace vectoriel R^{2n} . Une *structure presque complexe* sur V_{2n} sera définie par la donnée dans T_x d'une structure vectorielle complexe subordonnée à sa structure vectorielle réelle et dépendant d'une façon continue de x . Toute structure analytique complexe subordonnée à la structure différentiable de V_{2n} détermine sur V_{2n} une structure presque complexe, mais en général une structure presque complexe ne dérive pas d'une structure analytique complexe et on ignore si une *variété presque complexe* (c'est-à-dire, une variété munie d'une structure presque complexe) admet aussi une structure analytique complexe. La recherche des structures presque complexes sur V_{2n} est un problème de la théorie des espaces fibrés. Je rappellerai, en les complétant, les résultats que j'ai exposés au Colloque de Topologie Algébrique de Paris (1947) et j'indiquerai quelques résultats de Wen-tsün Wu; mais je ne pourrai pas exposer les méthodes de H. Hopf, qui a abordé la même question d'un point de vue un peu différent. Les nombres entre crochets renvoient à la bibliographie.

2. Structures fibrées subordonnées à une structure fibrée vectorielle [2]. Soit $E(B, F, G, H)$ un espace fibré de base B , de fibres isomorphes à F , de groupe structural topologique G . Nous supposons F muni d'une structure admettant G comme groupe d'automorphismes. Par les homéomorphismes distingués, dont l'ensemble est H , cette structure est transportée sur une structure bien déterminée dans chaque fibre F_x . H est alors l'ensemble des isomorphismes de F sur les fibres. Il est muni d'une structure fibrée $H(B, G, G_\gamma, \bar{H})$ et s'appelle *espace fibré principal associé*. Etant donné un sous-groupe G' de G , muni de la topologie induite, une structure fibrée $E(B, F, G', H')$ est dite *subordonnée* à $E(B, F, G, H)$ lorsque $H' \subset H$. Toute structure $E(B, F, G', H')$ détermine canoniquement une structure $E(B, F, G, H)$ à laquelle elle est subordonnée. Supposons que les classes sG' déterminent une structure fibrée sur G . Les structures $E(B, F, G', H')$ subordonnées à une structure donnée $E(B, F, G, H)$ correspondent alors d'une façon biunivoque aux *sections* de l'espace fibré associé à $E(B, F, G, H)$ par l'homomorphisme φ de G sur le groupe de transformations de l'espace homogène G/G' , défini par $\varphi(s)(tG') = s(tG')$. C'est l'espace H/G' des classes hG' , où $h \in G$; sa base est B et ses fibres sont isomorphes à G/G' . Les structures subordonnées $E(B, F, G', H')$ se répartissent en classes d'homotopie correspondant aux classes d'homotopie des sections; les structures d'une même classe sont iso-

morphes. Si $\mathfrak{S}'$ désigne une structure sur F admettant G' comme groupe d'automorphismes, l'espace H/G' peut s'appeler l'espace des structures isomorphes à $\mathfrak{S}'$ et subordonnées aux structures données sur les fibres F_x .

Soit R^n (resp. C^n, Q^n) l'espace numérique réel (resp. complexe, quaternionien), L_n (resp. L'_n, L''_n) le groupe linéaire homogène de R^n (resp. C^n, Q^n), O_n (resp. O'_n, O''_n) le groupe orthogonal dans R^n (resp. unitaire dans C^n , unitaire quaternionien dans Q^n), L_n^+ (resp. O_n^+) la composante connexe de l'unité de L_n (resp. O_n). Une structure $E(B, F, G, H)$ sera appelée structure fibrée *vectorielle réelle* (resp. *réelle orientée*, *complexe*, *quaternionienne*, *euclidienne*, *euclidienne orientée*, *hermitienne*, *hermitienne quaternionienne*) si G est L_n (resp. $L_n^+, L'_n, L''_n, O_n, O_n^+, O'_n, O''_n$), F étant suivant les cas R^n, C^n ou Q^n . Comme L_n/O_n (resp. $L_n^+/O_n^+, L'_n/O'_n, L''_n/O''_n$) est homéomorphe à un espace numérique, toute structure fibrée vectorielle réelle (resp. réelle orientée, complexe, quaternionienne) admet des structures euclidiennes (resp. euclidiennes orientées, hermitiennes, hermitiennes quaternioniennes) subordonnées et celles-ci appartiennent toutes à une même classe. En identifiant R^{2n} à C^n et C^n à Q^m , si $n = 2m$, on a $L'_n \subset L_{2n}^+, L''_m \subset L'_n, O'_n \subset O_{2n}^+, O''_m \subset O'_n$, et le problème d'existence de structures subordonnées se pose pour des structures du type suivant:

$$\begin{array}{ll} E(B, R^{2n}, L_{2n}, H), & E(B, R^{2n}, O_{2n}, \cdot) \\ E(B, R^{2n}, L_{2n}^+, \cdot), & E(B, R^{2n}, O_{2n}^+, \cdot) \\ E(B, C^n, L'_n, \cdot), & E(B, C^n, O'_n, \cdot) \\ E(B, Q^m, L''_m, \cdot), & E(B, Q^m, O''_m, \cdot). \end{array}$$

D'après la remarque précédente, on peut toujours se ramener au cas de structures figurant dans la deuxième colonne ci-dessus. L'espace $T(V_{2n})$ associé à une variété différentiable V_{2n} est muni d'une structure fibrée vectorielle réelle, dont les structures subordonnées s'appellent respectivement: structure *vectorielle tangente orientée*, *presque complexe*, *presque quaternionienne*, *riemannienne*, *presque hermitienne*, *presque hermitienne quaternionienne*.

3. Les structures vectorielles complexes sur R_{2n} . Soit $(\epsilon_1, \dots, \epsilon_n)$ la base canonique de C^n et identifions C^n à R^{2n} en identifiant $(\epsilon_1, i\epsilon_1, \dots, \epsilon_n, i\epsilon_n)$ à la base canonique de R^{2n} . Alors L'_n est le sous-groupe de L_{2n} qui laisse invariante la transformation I_0 définie par $I_0 z = iz$, où $z \in C^n$. Une structure vectorielle complexe sur R^{2n} , subordonnée à la structure vectorielle réelle, est définie par une transformation linéaire I de R^{2n} telle que $I^2 = -1$, c'est-à-dire, $I(Ix) = -x$ pour $x \in R^{2n}$; le produit du nombre complexe $a + bi$ par x sera $(a + bI)x$. Les vecteurs x et Ix sont linéairement indépendants et déterminent un plan invariant par I . R^{2n} admet des bases de la forme $(e_1, Ie_1, \dots, e_n, Ie_n)$; l'orientation correspondante de R^{2n} ne dépend que de I ; c'est l'*orientation associée à I* . L'espace des structures vectorielles complexes sur R^{2n} est L_{2n}/L'_n , dont la composante connexe L_{2n}^+/L'_n est l'espace des structures complexes dont l'orientation associée est aussi associée à I_0 .

Considérons R^{2n} comme l'espace des vecteurs réels de C^{2n} . La transformation I se prolonge à C^{2n} et admet les valeurs propres $\pm i$. L'ensemble des vecteurs propres correspondant à $-i$ forme un sous-espace X_n de dimension n . L'ensemble des vecteurs propres correspondant à $+i$ est $\bar{X}_n$, imaginaire conjugué de X_n , et l'on a $X_n \cap \bar{X}_n = 0$. Inversement tout sous-espace X_n de C^{2n} tel que $X_n \cap \bar{X}_n = 0$ détermine une transformation I . On peut définir X_n par n formes linéaires sur C^{2n} dont les restrictions à R^{2n} sont des formes linéaires à valeurs complexes ne s'annulant simultanément que pour le vecteur 0.

Posons $F(x, x) = x_1^2 + y_1^2 + \cdots + x_n^2 + y_n^2$, où $(x_1, y_1, \cdots, x_n, y_n)$ sont les coordonnées canoniques de $x \in R^{2n}$. O_{2n}/O'_n est l'espace des transformations I laissant invariante la forme quadratique $F(x, x)$, condition équivalente à $F(x, Ix) = 0$ ou $I \in O_{2n}^+$. L'espace X_n associé à I est alors une génératrice du cône défini dans C^{2n} par $F(x, x) = 0$. Donc O_{2n}^+/O'_n , que nous désignons par Γ_n , s'identifie à l'une des composantes connexes de l'espace des génératrices X_n de ce cône.

Comme $\lambda \in L_{2n}$ se prolonge à C^{2n} , l'espace fibré $T(V_{2n})$ admet un espace fibré associé $T^c(V_{2n})$ dont les fibres T_x^c sont isomorphes à C^{2n} et qui admet $T(V_{2n})$ comme sous-espace. Un élément de T_x^c s'appelle *vecteur complexe tangent* à V_{2n} en x , un sous-espace X_p de T_x^c s'appelle p -élément complexe tangent en x .

Une structure presque complexe sur V_{2n} est donc déterminée par un champ de transformations linéaires I , définies dans T_x et telles que $I^2 = -1$, ou par un champ de n -éléments complexes X_n tels qu'en chaque point $x \in V_{2n}$ on ait $X_n \cap \bar{X}_n = x$. Au voisinage de x elle est définie encore par n formes de Pfaff sur V_{2n} , à valeurs complexes et ne s'annulant simultanément pour aucun vecteur réel non nul. Il lui correspond une orientation bien déterminée de V_{2n} . Une structure presque hermitienne subordonnée à une structure riemannienne est définie par un champ de transformations orthogonales I ou par un champ de n -éléments isotropes.

4. Formes différentielles extérieures quadratiques sur V_{2n} . A une transformation I dans R^{2n} telle que $F(x, Ix) = 0$ est associée la forme bilinéaire alternée $\Psi(x, x') = F(Ix, x')$ de rang $2n$, et la forme $\Phi(x, x') = F(x, x') - i\Psi(x, x')$, qui est une forme d'Hermite définie positive par rapport à la structure complexe définie par I . A toute structure presque hermitienne sur V_{2n} est donc associée une forme différentielle extérieure quadratique Ω de rang $2n$; l'orientation associée est définie par Ω^n , forme de degré $2n$ non nulle en chaque point. Réciproquement toute forme différentielle extérieure quadratique Ω partout de rang $2n$ sur V_{2n} est associée de cette façon à des structures presque hermitiennes, qui sont toutes de même classe et qui correspondent à l'orientation définie par Ω^n . Ceci résulte du fait que $\bar{L}_{2n}/L'_n$ est homéomorphe à un espace numérique, $\bar{L}_{2n}$ désignant le sous-groupe de L_{2n} qui laisse invariante la forme $\Psi_0(x, x') = x_1 y'_1 - x'_1 y_1 + \cdots + x_n y'_n - x'_n y_n$.

Donc l'existence d'une structure presque complexe sur la variété orientée V_{2n} est équivalente à l'existence d'une forme différentielle extérieure quadratique

Ω telle que Ω^n soit non nulle partout et définisse l'orientation donnée. Appelons *variété presque kählérienne* une variété presque hermitienne dont la forme extérieure associée Ω est fermée, c'est-à-dire, $d\Omega = 0$. Appelons *variété symplectique* une variété V_{2n} munie d'une forme fermée Ω telle que $\Omega^n \neq 0$ en chaque point. Une variété symplectique admet toujours une structure presque kählérienne subordonnée et possède des propriétés topologiques plus particulières qu'une variété presque complexe quelconque. En particulier, si elle est compacte, ses nombres de Betti de dimension paire sont différents de 0, car $\Omega^k \sim 0$ pour $0 < k \leq n$ (remarque que je dois à G. de Rham).

5. Topologie de l'espace $\Gamma_n = O_{2n}^+/O_n'$. Γ_1 est un point. Γ_2 est homéomorphe à S_2 . Γ_3 est homéomorphe à l'espace projectif complexe $P_3(C)$. Γ_4 est homéomorphe à la quadrique complexe $Q_6(C)$ à 6 dimensions complexes. Quel que soit $n > 1$, Γ_n admet une structure fibrée de base S_{2n-2} et de fibre Γ_{n-1} . On en déduit les premiers groupes d'homotopie de Γ_n . $\pi_2(\Gamma_n) \cong \pi_2(\Gamma_2)$, cyclique infini, pour $n \geq 2$. Pour $i > 2$, on a $\pi_i(\Gamma_3) \cong \pi_i(S_7)$. Comme $Q_6(C)$ admet une structure fibrée [3] de base S_6 et de fibre $P_3(C)$, pour laquelle il existe une section¹, on a $\pi_i(\Gamma_4) \cong \pi_i(S_6) \times \pi_i(P_3(C))$.

Pour $n \geq 4$, on a $\pi_i(\Gamma_n) = 0$, si $2 < i < 6$, et $\pi_6(\Gamma_n)$ est cyclique infini. Ces propriétés de Γ_n servent à démontrer les résultats du §6.

6. Conditions d'existence de structures presque complexes.² Etant donné $E(B, R^{2n}, O_{2n}^+, H)$, le premier obstacle à l'existence d'une structure fibrée hermitienne subordonnée c'est-à-dire d'une section de l'espace fibré associé de fibres isomorphes à Γ_n , est une classe de cohomologie W^3 de B à coefficients entiers. Pour qu'il existe une section sur le squelette de dimension 3 de B , qui est supposé être un complexe, il faut et il suffit que $W^3 = 0$. La classe W^3 est identique à la classe caractéristique de Stiefel-Whitney, premier obstacle à l'existence d'un champ associant à tout $x \in B$ une suite de $2n - 2$ vecteurs indépendants de la fibre R_x^{2n} . La variété de Stiefel $V_{2n, 2n-2}$ est en effet un espace fibré de base Γ_n et de fibre $W_{n, n-1}$, variété des suites orthonormées de $n - 1$ vecteurs unitaires de l'espace hermitien C^n . Il en résulte un isomorphisme canonique de $\pi_2(\Gamma_n)$ sur $\pi_2(V_{2n, 2n-2})$ et l'identification des deux premiers obstacles considérés.

Si $W^3 = 0$, il y a un deuxième obstacle de dimension 4 pour $n = 2$, de dimension 8 pour $n = 3$, de dimension 7 pour $n > 3$. Si $n \geq 3$, $W^3 = 0$ entraîne donc $W^5 = 0$, où W^5 est la classe de Stiefel-Whitney de dimension 5. En tenant compte d'un isomorphisme canonique de $\pi_6(\Gamma_n)$ sur $\pi_6(V_{2n, 2n-6})$, où $n > 3$, on voit que le deuxième obstacle est identique à la classe W^7 de Stiefel-Whitney, premier obstacle à l'existence d'un champ de $2n - 6$ vecteurs. Une condition nécessaire pour l'existence d'une structure fibrée hermitienne subordonnée est évidemment que toutes les classes W^{2k+1} de Stiefel-Whitney soient nulles.

¹ $Q_6(C)$ n'est pas homéomorphe à $S_6 \times P_3(C)$, contrairement à ce que j'ai affirmé dans [2].

² On trouvera des résultats concernant les structures presque quaternioniennes dans [2] et [9].

En particulier soit $E = T(V_{2n})$, muni de la structure fibrée tangente orientée de la variété orientée V_{2n} . La classe W^3 de V_{2n} est le premier obstacle à l'existence d'une structure presque complexe sur V_{2n} . La condition $W^3 = 0$ est nécessaire et suffisante pour l'existence d'une structure presque complexe sur une variété orientée V_6 . Si de plus le groupe de cohomologie de dimension 2 de V_6 est nul, toutes les structures presque complexes sur V_6 et correspondant à une orientation donnée forment une seule classe. A chaque orientation de la sphère S_6 correspondent ainsi des structures presque complexes, appartenant toutes à une même classe d'homotopie. Mais il faudrait sans doute des méthodes nouvelles pour décider si S_6 admet aussi des structures analytiques complexes.

Chacun des parallélismes classiques dans l'espace projectif réel P_7 correspond [3] sur $Q_6(C)$ à une structure fibrée dont les fibres sont des génératrices isomorphes à $P_3(C)$. En supposant que S_6 soit la partie réelle de $Q_6(C)$, cette structure fibrée définit justement une structure presque hermitienne sur S_6 . Celle-ci pourra aussi être définie à l'aide des octaves de Cayley. Elle ne dérive pas d'une structure analytique complexe.

Par une méthode s'appliquant aux sphères S_{2n} , j'ai montré que S_4 n'admet aucune structure presque complexe, résultat obtenu d'une manière différente par H. Hopf. Si S_{2n} est presque complexe, S_{2n+1} est parallélisable (Kirchhoff, [5]), ce qui entraîne le fait que S_{4n} n'admet aucune structure presque complexe.

D'après Whitney [7], la classe W^3 d'une variété orientée V_4 est nulle. Mais Wen-tsün Wu a montré que pour tout $n > 2$, il existe des variétés orientées V_{2n} dont la classe W^3 n'est pas nulle.

7. Quelques résultats de Wen-tsün Wu. Par l'étude approfondie des variétés de Grassmann réelles et complexes, Wen-tsün Wu a obtenu les relations suivantes entre les classes caractéristiques d'une structure fibrée vectorielle réelle orientée $\mathcal{F}$ et celles d'une structure fibrée vectorielle complexe subordonnée $\mathcal{F}'$: Relations de Wu: $W_2(\mathcal{F}, t) = C_2(\mathcal{F}', t)$; $\bar{W}_2(\mathcal{F}, t) = \bar{C}_2(\mathcal{F}', t)$; $P(\mathcal{F}, t) = C(\mathcal{F}', t) \cup C(\mathcal{F}', it)$; $\bar{P}(\mathcal{F}, t) = \bar{C}(\mathcal{F}', t) \cup \bar{C}(\mathcal{F}', it)$; $X^{2n}(\mathcal{F}) = (-1)^n C^{2n}(\mathcal{F}')$.

Dans ces formules on a posé: $W_2(\mathcal{F}, t) = \sum W_2^k(\mathcal{F}) t^k$, $\bar{W}_2(\mathcal{F}, t) = \sum \bar{W}_2^k(\mathcal{F}) t^k$, où $W_2^k(\mathcal{F})$ (resp. $\bar{W}_2^k(\mathcal{F})$) sont les classes (resp. classes duales) de Stiefel-Whitney réduites modulo 2. $P(\mathcal{F}, t) = \sum (-1)^k P^{4k}(\mathcal{F}) t^{4k}$, $\bar{P}(\mathcal{F}, t) = \sum \bar{P}^{4k}(\mathcal{F}) t^{4k}$, où $P^{4k}(\mathcal{F})$ (resp. $\bar{P}^{4k}(\mathcal{F})$) sont les classes (resp. classes duales) de Pontrjagin. $C(\mathcal{F}', t) = \sum C^{2k}(\mathcal{F}') t^{2k}$, $\bar{C}(\mathcal{F}', t) = \sum \bar{C}^{2k}(\mathcal{F}') t^{2k}$, où $C^{2k}(\mathcal{F}')$ (resp. $\bar{C}^{2k}(\mathcal{F}')$) sont les classes (resp. classes duales) de Chern de $\mathcal{F}'$; ces deux polynômes réduits modulo 2 sont désignés par $C_2(\mathcal{F}', t)$ et $\bar{C}_2(\mathcal{F}', t)$. $X^{2n}(\mathcal{F})$ est la classe caractéristique d'Euler-Poincaré. On trouvera la définition précise de ces classes dans la thèse de Wu.

THÉORÈME DE WU. Les classes d'isomorphie des structures presque complexes sur une variété orientée V_4 correspondent d'une façon biunivoque aux classes de cohomologie C^2 sur V_4 telles que:

$$W_2^2(V_4) = C_2^2 ; P^4(V_4) + 2X^4(V_4) = C^2 \cup C^2,$$

où C_2^2 désigne la classe déduite de C^2 par réduction modulo 2, $P^4(V_4)$ la classe de Pontrjagin, et $X^4(V_4)$ la classe d'Euler-Poincaré. C^2 sera la classe de Chern de la structure presque complexe correspondante.

Les relations de Wu entraînent que S_{4k} n'admet pas de structure presque complexe. Ce résultat est valable pour toute variété V_{4k} dont l'anneau de cohomologie est isomorphe à celui de S_{4k} et dont la classe de Pontrjagin P^{4k} est nulle.

8. Les sous-variétés d'une variété presque complexe V_{2n} . Un p -élément X_p de V_{2n} sera dit *complexe* lorsqu'il est invariant par la transformation I définie dans l'espace tangent T_x qui contient X_p ; il sera dit *réel* lorsqu'il rencontre son transformé par I au point x seulement. Une sous-variété V_p de V_{2n} sera dite *presque complexe* (resp. *réelle*) lorsque les p -éléments tangents à V_p sont complexes (resp. réels). Une sous-variété presque complexe est munie d'une structure presque complexe induite; celle-ci dérive d'une structure analytique complexe si V_{2n} est analytique complexe.

Soit V_n une sous-variété réelle de V_{2n} . L'espace fibré $T(V_n)$ admet un isomorphisme sur l'espace fibré $N(V_n)$ des vecteurs normaux à V_n , les points de V_n restant fixés. Réciproquement si cette condition est vérifiée pour une sous-variété V_n d'une variété quelconque V_{2n} , il existe dans un voisinage de V_n une structure presque complexe telle que V_n soit une sous-variété réelle. En particulier, le voisinage de la diagonale Δ de $V_n \times V_n$ admet une structure presque complexe telle que Δ soit une sous-variété réelle. Il admet même une structure analytique complexe telle que Δ soit une sous-variété analytique réelle.

La position d'une sous-variété réelle V_n dépend de la structure de $T(V_n)$. L'espace fibré $T^c(V_n)$ admet un isomorphisme canonique dans $T(V_{2n})$ muni de la structure fibrée complexe. Si V_n est déformable en un point de V_{2n} , $T^c(V_n)$ est isomorphe à $V_n \times C^n$. Pour toute variété V_n , Wu a démontré la relation suivante: $C(V_n, t) = P(V_n, t)$, où $C(V_n, t)$ désigne le "polynôme de Chern" de $T^c(V_n)$ et $P(V_n, t)$ le "polynôme de Pontrjagin" de $T(V_n)$ définis au §7. Si V_n est une sous-variété réelle de V_{2n} , $C(V_n, t)$ est la trace sur V_n de $C(V_{2n}, t)$, polynôme de Chern de $T(V_{2n})$ muni de la structure fibrée complexe.

Pour $n = 2$, la relation de Wu donne $C(V_2, t) = 1$, ce qui montre que pour toute variété V_2 l'espace $T^c(V_2)$ est isomorphe à $V_2 \times C^2$.

Soit V_n une sous-variété réelle de l'espace projectif complexe $P_n(C)$. D'après la relation de Wu, la trace sur V_n de $C^{4k+2}(P_n(C))$ est nulle; il en résulte que les cycles de dimension $4k + 2$ de V_n sont ~ 0 dans $P_n(C)$. Par contre pour tout $k \leq n$ les cycles de dimension $2k$ d'une sous-variété complexe ne sont pas tous ~ 0 dans $P_n(C)$; ce résultat est valable aussi pour toute sous-variété presque complexe d'une variété presque kählérienne.

Les notions et les résultats précédents s'étendent aux variétés plongées (V_p, f) dans V_{2n} , où f est une application différentiable régulière de V_p dans V_{2n} . Si V_{2n} est muni d'une forme différentielle extérieure Ω telle que $\Omega^n \neq 0$ en chaque

point, les variétés intégrales de Ω sont des variétés plongées réelles pour une certaine structure presque complexe de V_{2n} .

9. Problème d'équivalence de deux structures presque complexes. Etant données deux variétés presque complexes V_{2n} et $\bar{V}_{2n}$, une *équivalence* de l'une à l'autre est un homéomorphisme différentiable de V_{2n} sur $\bar{V}_{2n}$ dont le prolongement à $T(V_{2n})$ est un isomorphisme de $T(V_{2n})$ sur $T(\bar{V}_{2n})$ par rapport aux structures fibrées vectorielles complexes. Si f est une application différentiable régulière d'une variété W_{2n} dans V_{2n} , à la structure presque complexe sur V_{2n} correspond une structure presque complexe sur W_{2n} , appelée image réciproque par f de la première. Si celle-ci est définie localement par n formes de Pfaff complexes $\omega_1, \dots, \omega_n$, son image réciproque est définie par les formes $f^*(\omega_k)$. Le problème d'équivalence locale de deux structures presque complexes peut être traité par les méthodes de E. Cartan et il vient d'être étudié pour $n = 2$ par Paulette Libermann [6].

Pour qu'une structure presque complexe sur V_{2n} dérive d'une structure analytique complexe, il faut et il suffit qu'elle soit partout localement équivalente³ à la structure complexe naturelle sur C^n , qui est définie par les formes $dz_1, dz_2, \dots, dz_n$. Soit g une équivalence d'un ensemble ouvert de V_{2n} à un ensemble ouvert de C^n . Les formes $dg_h = g^*(dz_h)$ sont alors des combinaisons linéaires indépendantes des formes ω_h , d'où l'on déduit les relations: $d\omega_h = \sum \omega_h \wedge \omega_{hl}$, où ω_{hl} sont des formes de Pfaff complexes sur V_{2n} . Ces équations, indiquées par G. de Rham, sont des conditions nécessaires pour que la structure presque complexe dérive d'une structure complexe. Dans le cas où les composantes réelles et imaginaires des formes ω_h sont des formes de Pfaff analytiques sur V_{2n} , ces conditions sont aussi suffisantes.⁴ En général, on aura:

$$d\omega_h = \sum \omega_h \wedge \omega_{hl} + \sum a_{hlm} \bar{\omega}_l \wedge \bar{\omega}_m.$$

On peut dire que les formes $\sum a_{hlm} \bar{\omega}_l \wedge \bar{\omega}_m$ définissent la torsion de la structure presque complexe; les a_{hlm} sont les composantes de son *tenseur hermitien de torsion*.

Remarquons que les formules intégrales de Chern [1] donnant les classes de Chern s'étendent au cas des structures presque complexes.

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UNIVERSITY OF STRASBOURG,
STRASBOURG, FRANCE.

COMPLEX-ANALYTIC MANIFOLDS¹

BENO ECKMANN

1. Generalities. 1.1. The concept of a *complex-analytic manifold* (in short *complex manifold*) is the natural generalization of the concept of a Riemann surface in the abstract sense. A $2m$ -dimensional complex manifold M is a manifold of dimension $2m$ in the usual sense which is covered by a family of systems of local complex coordinates $z_1, \dots, z_m$ (instead of the usual $2m$ real coordinates) in such a way that the relation between two such complex coordinate systems in the intersection of their existence domains is given by complex-analytic functions. Since the coordinate transformation is one-to-one, the Jacobian of these functions is not equal to 0, and the square of its absolute value is easily seen to be equal to the Jacobian of the corresponding real coordinate transformation. A complex manifold M is therefore orientable and has a natural orientation. We shall restrict ourselves throughout to closed manifolds. The set of all local complex coordinate systems which are admissible, i.e., which may be added to the given family in accordance with the analyticity condition, is called the *complex structure* of M . Concepts like analytic function, analytic map, etc. have an invariant meaning with respect to the given complex structure (i.e., are independent of special coordinate systems used to describe them).

1.2. Algebraic varieties in a complex projective space P_n (of n complex, i.e., $2n$ real, dimensions) have a natural complex structure, as well as P_n itself, and are therefore complex manifolds—provided that there are no “singularities”. There exist, on the other hand, examples of complex manifolds which cannot be imbedded in a P_n (cf. 3.4). The concept of a complex manifold is therefore more general than that of an algebraic variety. It is probably also more general than that of a higher-dimensional Riemann surface in a concrete sense, defined as the space of all function elements of an algebroid function on some standard space S ; but such a statement depends of course on the space S and on uniformization possibilities.

1.3. It is well-known that all orientable surfaces admit complex structures. For higher-dimensional manifolds (orientable, of even dimension) this is not the case. On the 4-dimensional sphere S^4 , for example, it is not possible to define a complex structure (cf. Hopf [1]², Ehresmann [2]). The general *existence problem* of a complex structure on a given manifold M , or the problem of what special properties of M are implied by such a structure, has several quite different aspects. Actually there is not much known about the complex structure itself; all consequences are deduced from assumptions which are weaker—the *almost complex* structure, or stronger—the existence of a *Kähler metric* (or Hermitean metric without torsion). The reason for these two approaches is simple: Almost complex structure is an

¹ This address was listed on the printed program under the title *Topologie der komplexen Mannigfaltigkeiten*.

² Numbers in brackets refer to the bibliography at the end of the address.

assumption concerning the tangent bundle of M , and therefore suitable for fiber space methods; and Kaehler metric is an assumption on the Riemannian or Hermitean geometry of M , which can be investigated by the methods of harmonic differential forms and of differential geometry. In both cases powerful existing theories can be applied.

Concerning the first approach I shall make only a few remarks, which overlap to some extent with Ehresmann's address; after that I shall discuss in more detail the second approach.

2. Almost complex manifolds. 2.1. A manifold M is called *almost complex* if for each point x of M a linear map I_x of the tangent space T_x at x into itself is defined such that $I_x^2 = -1$ ($1 =$ identity map) for all x , and such that I_x depends continuously upon x . This is possible only if M is of even dimension $2m$. A complex manifold is almost complex; for if we use in the tangent space T_x complex vector components corresponding to local complex coordinates, multiplication of the vector components by $i = (-1)^{1/2}$ is a linear map of the required type, independent of the special coordinate system. I do not know of an example of an almost complex manifold which does not admit a complex structure, but probably there are such.

2.2. Let M be almost complex and $v(x)$ a continuous nonzero vector field on M with one possible singularity; then $I_x v(x) = w(x)$ is a second field with one possible singularity at the same point and such that $w(x)$ and $v(x)$ are independent everywhere else since $I_x^2 = -1$. Therefore there exists on M a 2-field with one isolated singularity, which is obviously of a quite special nature. This fact was used by Hopf [1] to prove that certain manifolds (the spheres S^4 and S^8 and many other examples) do not admit an almost complex structure, i.e., a field of transformations I_x ; a fortiori these are not complex manifolds. Another way (cf. Ehresmann [2]) to prove existence or nonexistence of an almost complex structure on a manifold M is to apply the obstruction theory of fiberbundles; the base space of the fibering to be used is M , and the fiber at a point x is the space of all tensors at x describing transformations I_x (or related to them). For the simplest manifolds, the *spheres*, both methods give only quite special results, and in this case the existence problem seems to be related in a peculiar way to other topological questions, as I would like to indicate by some remarks.

2.3. If on a *sphere* S^n ($n = 2m$) a field of transformations I_x is given, we may assume that for all tangent vectors v at all points x , $I_x v$ is orthogonal to v . We have then for all pairs of *orthogonal* unit vectors x, y in $(n + 1)$ -dimensional Euclidean space E^{n+1} a unit vector function $z = F(x, y)$, z orthogonal to both x and y , continuous in x and linear in y ; this function is obtained by considering x as a point of S^n and y as a tangent vector at x , and defining $z = I_x y$. It can be extended in an obvious way to a function $\bar{F}(x, y) = z$ of two *arbitrary* vectors in E^{n+1} such that z is orthogonal to x and y and of length $(x^2 y^2 - (x \cdot y)^2)^{1/2}$ ($x \cdot y$ is the scalar product of x and y , $x^2 = x \cdot x$); $\bar{F}$ may be called a "*vector product in*

E^{n+1} , (continuous in the first factor, linear in the second). Applying this to the usual complex structure on S^2 , we obtain a vector product in E^3 which is the usual one, bilinear in x, y . By fiber space methods using special homotopy groups, it can be proved that *for $n = 4k$ there exists no such vector product in E^{n+1} of the required type, not even in the more general sense of continuity in both factors.* The proof is a simple extension of the method I used previously [3] for the same purpose, but only for $n = 4$ and 8 ; to extend it to all $n = 4k$, one has to compute more of the homotopy groups involved, which can be done by straightforward deformations. From the argument above it follows that *there is no almost complex structure on the spheres S^{4k}* (not even if one would admit fields of transformations I_x which are only continuous, not linear, in the tangent spaces T_x , but such that $v(x)$ and $I_x v(x)$ are always independent—this is, by the way, the concept of a “manifold of type I” considered by Hopf [1]).

2.4. It is well-known [3] that a vector product in E^{n+1} leads in a natural way to a *multiplication* in E^{n+2} , i.e., a rule associating with any two vectors $X, Y \in E^{n+2}$ a product $U = X \circ Y$, with a unit and with the “norm product rule” $U^2 = X^2 \cdot Y^2$. Namely, let $X \in E^{n+2}$ be given by a pair (x_0, x) of a real number x_0 and a vector $x \in E^{n+1}$; then $X \circ Y = U = (u_0, u)$ is defined by $u_0 = x_0 y_0 - x \cdot y$, $u = x_0 y + y_0 x + \bar{F}(x, y)$. The rule $U^2 = X^2 \cdot Y^2$ is easily checked, and $E = (1, 0)$ is the unit. For the usual vector product in E^3 (i.e., the complex structure on S^2), this gives exactly the *quaternion* multiplication in E^4 . The Cayley numbers in E^8 are related in the same way to a (bilinear) vector product in E^7 , hence to an almost complex structure on S^6 . It is not known whether it corresponds to a complex structure on S^6 , nor indeed whether there is at all a complex to complex structure on S^6 . [Added in proof: It has been proved (cf. Eckmann and Frölicher, C. R. Acad. Sci. Paris vol. 232 (1951) p. 2284) that the almost complex structure on S^6 , derived from the Cayley numbers does not belong to a complex structure on S^6 by studying the “integrability conditions” relating almost complex structures.]

Furthermore, if $U = X \circ Y$ is a multiplication in E^{n+2} as described before, continuous in X and linear in Y , we have for each X with $X^2 = 1$ an orthogonal transformation $U(Y) = X \circ Y$ of E^{n+2} such that for $Y = (1, 0)$, $U(Y) = X$; i.e., we have a map Φ of S^{n+1} into the orthogonal group Ω_{n+2} of $n + 2$ variables, such that under the natural projection of Ω_{n+2} onto S^{n+1} the map Φ becomes the identity of S^{n+1} . It is easy to see that this map Φ is nothing else than a global parallelism on S^{n+1} (a system of $n + 1$ pairwise orthogonal tangent unit vector fields). Combining this with 2.3, we obtain a theorem discovered and proved first by Kirchhoff [4] in a different way: *An almost complex structure on S^n induces a global parallelism on S^{n+1} .* From this it follows again that on S^{4k} there is no almost complex structure (same result as above, here only in the sense of linear I_x).

2.5. In all preceding remarks the existence problem of an almost complex or complex structure is considered in terms of a definite differentiable structure on the manifold, and it is not clear from the method whether for another differentiable structure the answer would be the same.

It might be worthwhile to note here that the complex structure on a (differentiable) manifold, if it exists, is in general not unique. There are manifolds which admit several different complex structures—different in the sense of analytic equivalence. The 2-dimensional torus, for example, has infinitely many different conformal structures (and the analogue holds for higher-dimensional tori); for all of them, however, the corresponding complex structure of the universal covering is the same. Hirzebruch [5] discovered the interesting result that there exist also *simply connected* manifolds with infinitely many different complex structures; the simplest example is the Cartesian product of S^2 by itself.

3. Complex structure and homology. 3.1. To obtain relations between homology properties and complex structure, it seems natural to study the influence of the transformations I_x belonging to that structure on *exterior differential forms*. A differential form α of degree r in the manifold M (in short, an r -form) may be considered, at each point x of M , as a skew-symmetric multilinear function of r vectors in the tangent space T_x ; if these vectors are transformed by I_x , α becomes a new form denoted by $C\alpha$ (if not only real, but also complex forms, i.e., complex functions in T_x , are admitted, it is convenient to include in the definition of C the passage to the conjugate-complex form; an explicit expression of C is given in 3.2). Obviously the operator C verifies $CC\alpha = (-1)^r\alpha$ for all r -forms α , hence C is an isomorphism of the linear group Φ^r of all r -forms in M onto itself. In general $dC\alpha \neq C d\alpha$, and $\tilde{d} = C^{-1}dC$ is a differential operator from Φ^r to Φ^{r+1} which is not equal to d , but has some of the same properties (e.g., $\tilde{d}\tilde{d} = 0$). $\tilde{d}$ leads to de Rham cohomology groups $\tilde{H}^r$ isomorphic to the usual ones H^r based on d . By Hodge's theorem ([6], [7]) these groups may be replaced by those of *harmonic* r -forms (with respect to a Riemann metric given on M), which are easier to handle than cohomology classes; i.e., by forms α with $\Delta\alpha = 0$ or $\tilde{\Delta}\alpha = 0$, where Δ is the generalized Laplace-Beltrami operator used by de Rham [6], and $\tilde{\Delta} = C^{-1}\Delta C$. It seems interesting to study the relation between Δ and $\tilde{\Delta}$; here we shall do this only in the case when we assume a complex structure in the strict sense, or even more than that. Before discussing this, we have to make some preliminary remarks.

3.2. On a $2m$ -dimensional complex manifold M we can use, in the calculus of differential forms, instead of the $2m$ real differentials dx_i (or vector components in T_x) the *complex differentials* dz_j , $j = 1, \dots, m$, corresponding to admissible coordinate systems $z_1, \dots, z_m$ and their conjugates $d\bar{z}_j$. The coordinate transformations, say from $z_1, \dots, z_m$ to $\zeta_1, \dots, \zeta_m$ being analytic, the dz_j are carried over to the $d\zeta_j$ and the $d\bar{z}_j$ to the $d\bar{\zeta}_j$ only, this splitting of the $2m$ differentials into two groups has an invariant meaning. A differential form which is homogeneous with respect to the $d\bar{z}_j$ will be called *pure*; and the corresponding degree in the $d\bar{z}_j$ only, its *type*. Any r -form α has a unique decomposition as a sum of pure forms $\alpha = \alpha_{(0)} + \dots + \alpha_{(r)}$ of type $h = 0, 1, \dots, r$. For a form $\alpha_{(h)}$ of type h , $C\alpha_{(h)}$ is given by $(-1)^h i^r \bar{\alpha}_{(h)}$, which is of type $r - h$; for pure forms, $\tilde{d}$ differs therefore from d only by a constant factor.

3.3. The Riemann metric, used in the definition of Δ , is called *Hermitean* if in

the given complex structure the line element ds^2 may be written as $\sum_{j,k} h_{jk} (dz_j d\bar{z}_k)$, $h_{jk} = \bar{h}_{kj}$, where $(\dots)$ indicates the *ordinary* product of differentials. Such a metric is however not sufficient for our purposes. For it appears that, roughly speaking, one must, in the relations between harmonic forms and complex structure, consider permutations of the second covariant derivatives involved in Δ and therefore use the Riemann curvature tensor of the metric; and moreover, one should compute this tensor by the complex formalism, i.e., in the so-called Hermitean geometry. This is in general not possible: it is easy to see that the connexions (the Christoffel symbols) computed from the same Hermitean metric in Riemannian and in Hermitean geometry are *different*. There is a special case where they coincide, and this case seems of great importance: namely when the metric is a “*Kaehler metric*” [8]; i.e., has the special property that the exterior differential form $\omega = \sum_{j,k} h_{jk} dz_j d\bar{z}_k$ is *closed* ($d\omega = 0$). Actually many of the homology properties we are going to describe can be deduced from assumptions which are stated in terms of real differential geometry, without using a complex structure; these are however less natural and intuitive, and for the purpose of this address we shall stay within the frame of the Kaehler metric.

It is well-known that on the complex projective space P_n there exists a Kaehler metric, and therefore on all algebraic varieties imbedded without singularities in P_n . More generally, if a complex manifold M is imbedded analytically in a Kaehler manifold, the induced metric on M fulfills also the Kaehler condition (cf. 3.6).

3.4. In the case of a Kaehler metric, the explicit expression of Δ yields the following results (cf. [10]) on pure forms and on C . (a) If a form α is harmonic, then all the “pure components” $\alpha_{(h)}$ of α (cf. 3.2) are also harmonic. In other words: H^r is the direct sum $H^r_{(0)} + H^r_{(1)} + \dots + H^r_{(r)}$, where $H^r_{(h)}$ is the group of pure harmonic forms of type h . If $p^r_{(h)}$ denotes the rank of $H^r_{(h)}$, we have for the Betti number $p^r = \sum_{h=0}^r p^r_{(h)}$. Of course $p^r_{(h)}$ is also defined for a general Hermitean metric; but all one can prove without the Kaehler condition is the inequality $p^r \geq \sum p^r_{(h)}$. (b) $\tilde{\Delta} = \Delta$. Let $\tilde{H}^r_{(h)}$ be the analogue of $H^r_{(h)}$, computed from $\tilde{\Delta}$ instead of Δ , $\tilde{p}^r_{(h)}$ its rank; it follows that $\tilde{H}^r_{(h)} = H^r_{(h)}$ and $\tilde{p}^r_{(h)} = p^r_{(h)}$. But since by definition $C\tilde{\Delta} = \Delta C$, C maps $\tilde{H}^r_{(h)}$ isomorphically onto $H^r_{(r-h)}$; hence $\tilde{p}^r_{(h)} = p^r_{(r-h)} = p^r_{(h)}$. Therefore, if r is odd, $p^r = 2\sum_{h=0}^{(r-1)/2} p^r_{(h)}$, i.e., $p^r \equiv 0 \pmod{2}$; if r is even, $p^r \equiv p^r_{(r/2)} \pmod{2}$. For algebraic varieties the fact that p^r is even in odd dimensions r is a well-known result already proved by Lefschetz and, using differential forms, by Hodge.

There exist complex manifolds which do not admit a Kaehler metric since their Betti numbers do not agree with the above conditions, namely the manifolds of topological structure $S^1 \times S^{2k+1}$ discovered by Hopf [1] (a complex structure on $S^1 \times S^3$, e.g., is obtained from the covering of $S^1 \times S^3$ by E^4 without the origin). From this it follows that these complex manifolds cannot be imbedded analytically in a complex projective space (nor in any Kaehler manifold).³

³ Added in proof: E. Calabi and the author showed that the product of any two odd-

3.5. All this is only one, somewhat special, aspect of the Kaehler metric, and there are other surprisingly strong consequences.

There is one immediate fact, which does not even use harmonic forms, but just some simple properties of the closed 2-form ω (cf. 3.3), depending only weakly upon the relation of ω to the metric. Let $n = 2m$ be the dimension of the Kaehler manifold M . The m th power ω^m is equal, up to a constant factor not equal to 0, to $|h_{jk}| dz_1 d\bar{z}_1 \cdots dz_m d\bar{z}_m$; since the determinant $|h_{jk}|$ is everywhere positive, $\int_M \omega^m \neq 0$ (ω^m is actually, except for a constant nonzero factor, the volume element of the metric considered as a Riemann metric). Hence ω^m is not cohomologous to 0, and the same is true for all powers ω^k , $k = 0, 1, \dots, m$; the corresponding Betti numbers p^{2k} are therefore greater than or equal to 1.

These nonvanishing cohomology classes in even dimensions may be carried over by the duality operator D of M to homology classes. Using the same symbol α for a closed form and for the corresponding cohomology class, the dual homology class $D\alpha$ is given by $\alpha \cap M$ (here M is the fundamental $2m$ -cycle of the oriented manifold, and $\cap$ the Čech-Whitney cap-product). Let us call the classes $D\omega^k = \omega^k \cap M = Z_{2(m-k)}$ of dimension $2(m-k)$ the *principal homology classes* of M . Clearly, writing Z for $Z_{2(m-1)} = D\omega$, $Z_{2(m-2)}$ is the *intersection* $Z \otimes Z$, $Z_{2(m-3)} = Z \otimes Z \otimes Z$, etc. If M is the complex projective space P_m , the principal classes Z_{2q} , $q = 0, 1, \dots, m$, are all represented, up to certain numerical factors, by the projective planes P_q in P_m .

3.6. Let L' be a complex manifold of dimension $2l < 2m$, and f an *analytic* and locally topological *imbedding* of L' in the Kaehler manifold M , $f(L') = L \subset M$. Considering the induced dual homomorphism f^* , the image $f^*\omega$ is the form ω' in L' corresponding to the induced metric in L' , which obviously is also a Kaehler metric. Thus, by the same argument as above, $0 \neq \int_{L'} \omega'^l = \int_{L'} f^*\omega^l = \int_L \omega^l$; by Stokes theorem it follows that $L \not\sim 0$ (not homologous to 0) in M .

Furthermore consider the principal homology classes $Z'_{2(l-k)}$ of L' ($k = 0, \dots, l$), and their images $f(Z'_{2(l-k)})$ in M ; they are equal to $f(\omega'^k \cap L') = f(f^*\omega^k \cap L') = \omega^k \cap f(L') = \omega^k \cap L = Z_{2(m-k)} \otimes L$, i.e., they are the intersections of L with the principal classes of M (for example, the principal classes of a manifold L imbedded in a complex projective space are given by the *plane sections* of L). These classes are all ~ 0 in M . For $\int_{\omega^k \cap L} \omega^{l-k} = \int_L \omega^l \neq 0$, hence by Stokes theorem $\omega^k \cap L \sim 0$.

Analogous remarks apply to “analytic cycles” in M , i.e., to the set of zeros of a distribution of local analytic functions. The problem of finding not only necessary, but also sufficient conditions for a homology class to contain analytic cycles seems to be difficult.

3.7. Very precise results on the role of ω and of principal classes in the homology structure of a Kaehler manifold M , including all of the known general homology structure of algebraic varieties, are obtained if, following the method of Hodge, harmonic forms in M are investigated in connection with special properties of

dimensional spheres $S^{2p+1} \times S^{2q+1}$ can be given a complex structure; for $p > 0$, $q > 0$ this is an example of a *simply connected* complex manifold which does not admit a Kaehler metric.

ω and of the differential geometry of the Kaehler metric. All these results are due to some simple formulas relating ω to the usual operations on differential forms: $d, *$ ($*\alpha$ is the adjoint of the r -form α , with respect to the metric, of degree $n - r = 2m - r$; cf. [6]), $\delta = \pm * d *$, $\Delta = d\delta + \delta d$, and the operator C , with $\tilde{d}, \tilde{\delta}$, etc. These formulas, most of which appear in Hodge's book [7] and in a note by A. Weil [9], have been completely established and discussed by Guggenheimer and myself [10]. They are of purely local character.

The main formula is a relation between $*$ and ω , replacing the operator $*$ for r -forms with $r \leq m$ by C and by multiplication with ω^{m-r} (and with a certain numerical factor u ; I omit here all details about such factors): $*\alpha = u\omega^{m-r}C\alpha$. This is however not true for all forms α . It holds (1) if $\omega * \alpha = 0$ —such forms are called *effective*; and more generally (2) if $\alpha = \omega^k\beta$, where β is effective—we shall call such a form *simple, of class k* . The nonzero factor u depends upon m, r , and k . Under condition (2) we have for $r \leq m - 2$, $\omega * (\omega\alpha) = u'\omega \cdot \omega^{m-r-2} C\omega\alpha = u'' * \alpha$, with nonzero factors u', u'' . From this it follows that (a) $\omega\alpha = 0$ only if $\alpha = 0$. By an easy induction argument it can be proved that (b) *every r -form α ($r \leq m$) has a unique decomposition into a sum of simple forms of class $k = 0, 1, \dots, q = [r/2]$, $\alpha = \beta_0 + \omega\beta_1 + \dots + \omega^q\beta_q$* . (a) holds therefore for all forms of degree $r \leq m - 2$: (c) *Multiplication by ω is an isomorphism of Φ^r , the group of all r -forms, into Φ^{r+2}* .

Further relations give $\Delta(\omega^k\alpha) = \omega^k(\Delta\alpha)$ for all forms α , and commutation of Δ with all operators involved. Hence all results apply in particular to harmonic forms, i.e., to cohomology groups H^r . From (b) above it follows that H^r is for $r \leq m$ the direct sum $H_0^r + \omega H_0^{r-2} + \dots + \omega^q H_0^{r-2q}$, where $H_0^{r-2k} \subset H^{r-2k}$ is the group of effective harmonic $(r - 2k)$ -forms; from (c), that multiplication by ω is, for $r \leq m - 2$, an *isomorphism of H^r into H^{r+2}* (cf. [7], Chap. IV). For the Betti numbers of a Kaehler manifold M we obtain therefore $p^r \leq p^{r+2}$ for $r \leq m - 2$ (and, of course, the dual statements for higher dimensions). These are strong topological conditions for the existence of a Kaehler metric; they are not fulfilled by all complex manifolds (e.g., by the examples mentioned in 3.4).

3.8. To translate the results from cohomology to homology, we take the operator D , which maps H^r isomorphically onto the r th homology group H_r of M (with complex coefficients). Since $*\omega^k = u\omega^{m-k}$, $D*\omega^k$ is, except for a constant nonzero factor, the principal homology class Z_{2k} of M , (e.g., a plane section if M lies analytically in a complex projective space). Each homology class z_r of M of dimension $r \leq m$ may be written as $z_r = D*\alpha = D*(\beta_0 + \omega\beta_1 + \dots + \omega^q\beta_q)$, $q = [r/2]$, according to (b) in 3.7, with uniquely determined effective harmonic forms $\beta_0, \dots, \beta_q$. The term $D*\omega^k\beta_k$ is, except for the nonzero factor u , equal to $D(C\omega^k\beta_k\omega^{m-r}) = C\omega^k\beta_k\omega^{m-r} \cap M$; that is, to $C\omega^k\beta_k \cap (\omega^{m-r} \cap M) = C\omega^k\beta_k \cap D\omega^{m-r} = C\omega^k\beta_k \cap Z_{2r} = DC\omega^k\beta_k \otimes Z_{2r}$, or to $C\beta_k \cap (\omega^{m-(r-k)} \cap M) = DC\beta_k \otimes Z_{2(r-k)}$. Therefore, (a) *all homology classes of dimension $r \leq m$ lie on the principal class Z_{2r} , in the sense that they are intersections of something with Z_{2r} , and (b) they have a unique decomposition into a sum $z_r = z_r^{[0]} + z_r^{[1]} + \dots + z_r^{[q]}$, where $z_r^{[k]} = D*\omega^k\beta_k$ lies on $Z_{2(r-k)}$ (but not on Z_{2l} , $l < r - k$). Using*

only the assumption of a Kaehler metric we obtain thus a geometric situation of the same nature as the homology structure of algebraic varieties discovered by Lefschetz.

3.9. I would like to conclude with a remark on the differential forms themselves. If we have a simple form $\alpha = \omega^k \beta$ (cf. 3.7.), and if α is closed, it follows easily from the general relations that $\bar{\delta}\alpha = 0$; if in addition α is pure, this means $\delta\alpha = 0$, hence α is harmonic. A closed form α which is pure and simple is therefore always harmonic. If in particular $\alpha = d\gamma$, it must be equal to 0 (here we use a global result, based on the fact that M is closed), and we have the theorem: *If on a Kaehler manifold a form α is such that α and $d\alpha$ are both pure and simple, then $d\alpha = 0$, and α is harmonic.*

Let us take for example an *analytic* form α , i.e., a form in the dz_j only, with analytic coefficients. α is of type 0, and also $d\alpha$, since for all coefficients a of α we have $\partial a / \partial \bar{z}_j = 0$; it is easily seen that a form of type 0 is always effective. Therefore, by the above theorem, for an analytic differential form α in a Kaehler manifold, $d\alpha$ is always equal to 0, and α harmonic. Relations between the set of all analytic differential forms α in the manifold and its Betti numbers (cf. [7], [9], [10]) are therefore obtained without assuming in advance $d\alpha = 0$.

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EIDGENÖSSISCHE TECHNISCHE HOCHSCHULE,
ZÜRICH, SWITZERLAND.

COHOMOLOGY ON REAL DIFFERENTIABLE MANIFOLDS

CARL B. ALLENDOERFER

1. Introduction. The theory of de Rham has established a relationship between the cohomology theory (real coefficients) of a real, closed differentiable manifold M_n and the theory of exterior differential forms on this manifold. The connection is expressed by the theorems:

FIRST THEOREM OF DE RHAM. *Let C^r be a fixed r -dimensional cohomology class (real coefficients) on M_n and let Z_r be an arbitrary homology class (real coefficients) on M_n . Then there exists a regular closed r -form Θ^r ($d\Theta^r = 0$) such that:*

$$C^r \cdot Z_r = \int_z \Theta^r$$

where z is any r -cycle representing Z_r .

SECOND THEOREM OF DE RHAM. *If for a closed r -form Θ^r*

$$\int_z \Theta^r = 0$$

for all r -cycles (real coefficients) z , then there exists a regular $r - 1$ form Ψ^{r-1} on M_n such that:

$$d\Psi^{r-1} = \Theta^r;$$

i.e., Θ^r is derived.

This theory, however, is inadequate for a treatment of the relationship between differential forms and cohomology with coefficients which are integers or integers mod p (p a prime). It has been suggested by A. Weil [4] that such a theory can be constructed using singular r -forms. A specific example of such a theory was given by the author [1] in connection with the characteristic classes of a Riemannian manifold, and has served as motivation for the present paper which presents a general theory of this nature. In addition, the results are applied to harmonic forms and give a full cohomology theory for these forms which is an extension of the work of Hodge [2] and Kodaira [3]. The theory is also applied to derive an extension of de Rham's second theorem for cases in which the periods are not all zero (Theorem 6).

2. Elementary forms. Let M_n be a real, closed, differentiable manifold of class greater than or equal to 3. Consider a differentiable triangulation of M_n whose cells of dimension r are E_r^i ($i = 1, 2, 3, \dots$), and whose r -skeleton is K_r . Also consider a dual, differentiable subdivision of M_n whose $n - r$ skeleton is $*K^{n-r}$. Let C^r be a cochain on K_r whose values on E_r^i are α_i , where α_i are real

numbers. Associated with C^r is a dual chain $*C^{n-r}$ of $*K^{n-r}$ such that the intersection number $E_r^i \cdot *C^{n-r} = \alpha_i$. We prove the following theorem.

THEOREM 1. *There exist differential forms*

- (1) θ^r defined on K_r and regular of class greater than or equal to 1 on K_r ;
 (2) φ^{r-1} defined on K_r and regular of class greater than or equal to 2 on K_{r-1} with at most isolated point singularities on $K_r - K_{r-1}$, such that

$$(3) \quad d\varphi^{r-1} = \theta^r;$$

$$(4) \quad \int_{E_r^i} \theta^r - \int_{\partial E_r^i} \varphi^{r-1} = \alpha_i.$$

PROOF. Choose a cell E_r^i . We seek to define forms θ_i^r and φ_i^{r-1} having properties (1), (2), and (3) of the theorem and such that

$$(5) \quad \int_{E_r^j} \theta_i^r - \int_{\partial E_r^j} \varphi_i^{r-1} = \delta_{ij}.$$

There are numerous ways of finding forms $*\theta_i^r$ and $*\varphi_i^{r-1}$ having properties (1), (2), and (3) and such that $*\varphi_i^{r-1}$ is regular on $[K_r \cap \text{Star } E_r^i - E_r^i]$ and

$$(6) \quad \int_{E_r^i} *\theta_i^r - \int_{\partial E_r^i} *\varphi_i^{r-1} = 1,$$

but which do not necessarily satisfy (5) for E_r^j ($j \neq i$). Examples of such forms appear in the author's paper [1], but much simpler ones can be constructed.

Let U_n and V_n be neighborhoods of M_n such that $\text{Star } E_r^i \supset V_n \supset U_n \supset E_r^i$. Let f be a "local function", regular of class greater than or equal to 2 in M_n such that

$$f = \begin{cases} 1 & \text{in } U_n \\ 0 & \text{in } M_n - V_n. \end{cases}$$

Define $\varphi_i^{r-1} = f \cdot *\varphi_i^{r-1}$ on K_r ;

$$\theta_i^r = \begin{cases} *\theta_i^r & \text{on } E_r^i \\ d\varphi_i^{r-1} & \text{on } K_r - E_r^i. \end{cases}$$

These forms have properties (1), (2), (3), and (5). Now define

$$\theta^r = \sum_i \alpha_i \theta_i^r; \quad \varphi^{r-1} = \sum_i \alpha_i \varphi_i^{r-1}.$$

These forms have the required properties (1), (2), (3), and (4). Note that φ^{r-1} is regular on those cells for which $\alpha_i = 0$.

COROLLARY. *If B_r is any r -chain (real coefficients) of K_r , the forms of Theorem 1 are such that*

$$\int_{B_r} \theta^r - \int_{\partial B_r} \varphi^{r-1} = C^r \cdot B_r.$$

3. Extensions of elementary forms. In extending the forms θ^r and φ^{r-1} of Theorem 1 over the whole of M_n we shall proceed by induction from K_r to K_{r+1} to $\cdots$ to K_{n-1} to $K_n = M_n$. In extending over any cell E_{t+1} of K_{t+1} ($t = r, \cdots, n-1$) we shall use one of three methods listed below according to the circumstances involved. We call the extended forms: Θ^r and Φ^{r-1} respectively.

The methods described below make use of a special "polar" coordinate system in E_{t+1} . Let $x^1, \cdots, x^t$ be local coordinates on ∂E_{t+1} . Let L^{t-r} be a subcomplex of $\partial E_{t+1} \cap *K^{n-r}$ which will be chosen to meet particular needs in the sequel. In E_{t+1} there is a vertex P of $*K^{n-r}$ and a subcomplex of $*K^{n-r}$, L^{t-r+1} , which contains P and is such that $L^{t-r+1} \cap \partial E_{t+1} = L^{t-r}$. Join P to ∂E_{t+1} with a simple differentiable family of curves such that the curves from P to L^{t-r} lie on L^{t-r+1} . Parametrize these curves with the parameter x^{t+1} so that $x^{t+1} = 0$ is at P and $x^{t+1} = 1$ is on ∂E_{t+1} . The set $(x^1, \cdots, x^{t+1})$ are our local "polar" coordinates for E_{t+1} .

METHOD 1. On ∂E_{t+1} assume Θ^r regular and closed, Φ^{r-1} regular, and $d\Phi^{r-1} = \Theta^r$. Further if $t = r$, assume $\int_{\partial E_{r+1}} \Theta^r = 0$.

Then by the second theorem of de Rham there exists a regular form ψ^{r-1} on ∂E_{t+1} such that $d\psi^{r-1} = \Theta^r$. Extend ψ^{r-1} differentiably of class greater than or equal to 2 over E_{t+1} so that in a neighborhood of ∂E_{t+1}

$$(1) \quad \begin{aligned} \psi_{j_1 \cdots j_{r-1}}^{r-1}(x^1, \cdots, x^t, x^{t+1}) &= \psi_{j_1 \cdots j_{r-1}}^{r-1}(x^1, \cdots, x^t, 1) \text{ for } j_1 \cdots j_{r-1} \neq t+1; \\ \psi_{j_1 \cdots j_{r-2}t+1}^{r-1}(x^1, \cdots, x^{t+1}) &= 0. \end{aligned}$$

This extension is trivial for there are no other restrictions on ψ^{r-1} . Now define $\Theta^r|E_{t+1} = d\psi^{r-1}$. Then Θ^r is regular and closed on E_{t+1} and reduces to the given form on ∂E_{t+1} .

To extend Φ^{r-1} , define

$$(2) \quad \omega^{r-1} = \psi^{r-1} - \Phi^{r-1} \quad \text{on } \partial E_{t+1}.$$

Then ω^{r-1} is closed and regular on ∂E_{t+1} and can be extended regularly over E_{t+1} by the method just used for Θ^r . Finally define

$$(3) \quad \Phi^{r-1} = \psi^{r-1} - \omega^{r-1} \quad \text{on } E_{t+1}.$$

It follows that Φ^{r-1} is regular on E_{t+1} and that $d\Phi^{r-1} = \Theta^r$.

METHOD 2. On ∂E_{t+1} assume Θ^r as in Method 1, Φ^{r-1} singular on L^{t-r} , and $d\Phi^{r-1} = \Theta^r$ on $\partial E_{t+1} - L^{t-r}$.

Extend Θ^r as in Method 1. Define ω^{r-1} on $\partial E_{t+1} - L^{t-r}$ by (2) above. In our "polar" coordinate system define ω^{r-1} on $E_{t+1} - L^{t-r+1}$ by equations (1) above with ω substituted for ψ . It is clear that we still have $d\omega = 0$. Finally define Φ^{r-1} by (3) above. Φ^{r-1} is now regular on $E_{t+1} - L^{t-r+1}$ and where it is regular: $d\Phi^{r-1} = \Theta^r$.

METHOD 3. On ∂E_{t+1} assume Θ^r regular on $\partial E_{t+1} - \Gamma^{t-r-1}$, Φ^{r-1} regular on $\partial E_{t+1} - L^{t-r}$, Γ^{t-r-1} a subcomplex of L^{t-r} , and where defined $d\Phi^{r-1} = \Theta^r$.

We use the "polar" coordinate system and call Γ^{t-r} the subcomplex of L^{t-r+1} consisting of curves joining P to points of Γ^{t-r-1} . Then we extend Θ^r and Φ^{r-1} by the equations:

$$\begin{aligned} \text{On } E_{t+1} - \Gamma^{t-r} \quad & \left\{ \begin{aligned} \Theta_{j_1 \dots j_r}^r(x^1, \dots, x^t, x^{t+1}) &= \Theta_{j_1 \dots j_r}^r(x^1, \dots, x^t, 1) \\ &\text{for } j_1 \dots j_r \neq t+1; \\ \Theta_{j_1 \dots j_{r-1} t+1}^r(x^1, \dots, x^{t+1}) &= 0. \end{aligned} \right. \\ \text{On } E_{t+1} - L^{t-r+1} \quad & \left\{ \begin{aligned} \Phi_{j_1 \dots j_{r-1}}^{r-1}(x^1, \dots, x^t, x^{t+1}) &= \Phi(x^1, \dots, x^t, 1) \\ &\text{for } j_1 \dots j_{r-1} \neq t+1; \\ \Phi_{j_1 \dots j_{r-2} t+1}^{r-1}(x^1, \dots, x^{t+1}) &= 0. \end{aligned} \right. \end{aligned}$$

Then Θ^r and Φ^{r-1} are regular where defined and $d\Phi^{r-1} = \Theta^r$.

4. Cochains. We now use these methods of extension to derive the following consequences of Theorem 1.

THEOREM 2. *Given an r -cochain C^r (real coefficients) of K_r and its associated dual chain $*C^{n-r}$ of $*K^{n-r}$. There exist differential forms Θ^r and Φ^{r-1} on M_n such that*

- (1) Θ^r is regular of class greater than or equal to 1 and closed on $M_n - \partial *C^{n-r}$;
- (2) Φ^{r-1} is regular of class greater than or equal to 2 on $M_n - *C^{n-r}$;
- (3) $d\Phi^{r-1} = \Theta^r$ on $M_n - *C^{n-r}$;
- (4) $\int_{B_r} \Theta^r - \int_{\partial B_r} \Phi^{r-1} = C^r \cdot B_r$,

where B_r is any chain (real coefficients) of K_r .

PROOF. On K_r construct the forms θ^r and φ^{r-1} of Theorem 1. θ^r is regular on K_r and φ^{r-1} has point singularities on those cells E_i^r for which $\alpha_i \neq 0$. We can place these singularities on $*C^{n-r}$, for the intersection of $*C^{n-r}$ and such a cell is exactly one point (with a coefficient α_i).

In extending from K_t to K_{t+1} we have three situations for a cell E_{t+1} :

(1) $\partial E_{t+1} \cap *C^{n-r} = 0$. Thus Θ^r and Φ^{r-1} are regular on ∂E_{t+1} and when $t = r$, $\int_{\partial E_{r+1}} \Theta^r = 0$. Apply Method 1 above and obtain regular extensions of both forms over E_{t+1} .

(2) $\partial E_{t+1} \cap *C^{n-r} \neq 0$, but $E_{t+1} \cap \partial *C^{n-r} = 0$. Then Θ^r is regular on ∂E_{t+1} and when $t = r$, $\int_{\partial E_{r+1}} \Theta^r = 0$; but Φ^{r-1} has singularities on $\partial E_{t+1} \cap *C^{n-r}$ which we now call L^{t-r} . Apply Method 2 above and obtain Θ^r regular over E_{t+1} and Φ^{r-1} regular over $E_{t+1} - L^{t-r+1}$. Note that the singular locus of φ^{r-1} , namely L^{t-r+1} , remains on $*C^{n-r}$.

(3) $\partial E_{t+1} \cap *C^{n-r} \neq 0$ and $E_{t+1} \cap \partial *C^{n-r} \neq 0$. Then Θ^r is regular on ∂E_{t+1} except on $\partial E_{t+1} \cap \partial *C^{n-r}$ which we now call Γ^{t-r-1} . When $t = r$, this intersection

is necessarily empty, but now $\int_{\partial E_{t+1}} \theta^r \neq 0$. Also Φ^{r-1} is regular on ∂E_{t+1} except on L^{t-r} as in Case 2. Apply Method 3 above and obtain θ^r regular over $E_{t+1} - \Gamma^{t-r}$ and Φ^{r-1} regular over $E_{t+1} - L^{t-r+1}$. When $t = r$, we define Γ^{t-r} to be the point P described above. Note that Γ^{t-r} lies on $\partial^* C^{n-r}$ and L^{t-r+1} on $*C^{n-r}$.

By carrying out this process successively for t running from r to $n - 1$, we establish the theorem.

COROLLARY. *Theorem 2 also applies to any chains B_r of M_n (not necessarily those of K_r) provided that $B_r \cap \partial^* C^{n-r} = 0$ and $\partial B_r \cap *C^{n-r} = 0$.*

5. Cocycles. When C^r is a cocycle (real coefficients), $\partial^* C^{n-r} = 0$ and θ^r is regular on M_n . This amounts to the following theorem.

THEOREM 3. *Given a cocycle C^r on M_n , there exist differential forms θ^r and Φ^{r-1} on M_n such that*

(1) θ^r is closed and regular of class greater than or equal to 1 on M_n , and conditions (2), (3), and (4) of Theorem 2 and its corollary are satisfied.

When B_r is an r -cycle (real coefficients), this theorem amounts to the first theorem of de Rham. Since an r -dimensional cohomology class (real coefficients) is determined by its periods over the r -cycles (real coefficients), there is nothing more to say. However, a cohomology class (integer coefficients) requires for its determination not only its periods on the integral cycles of M_n but also its values on those integral chains which are cycles mod p . Theorem 3 then gives as a supplement to the de Rham theorem for this case the following theorem.

THEOREM 4. *Let $C_{(0)}^r$ be a fixed r -dimensional cohomology class (integer coefficients) on M_n and $Z_r^{(0)}$ and $Z_r^{(p)}$ arbitrary homology classes with integer and integer mod p coefficients respectively on M_n . Then there exists a regular closed form θ^r and a form Φ^{r-1} with singularities on $*C_{(0)}^{n-r}$ (the dual cycle associated with a particular cocycle $C_{(0)}^r$ representing $C_{(0)}^r$) and with $d\Phi^{r-1} = \theta^r$ on $M_n - *C_{(0)}^{n-r}$ such that*

$$C_{(0)}^r \cdot Z_r^{(0)} = \int_z \theta^r,$$

$$C_{(0)}^r \cdot Z_r^{(p)} \equiv \int_z \theta^r - \int_{\partial z} \Phi^{r-1} \pmod{p},$$

where z is a cycle which represents the corresponding homology class and on which the integrals are defined.

We can apply Theorem 2 to cochains $C_{(p)}^r$ with integers mod p as coefficients by selecting a particular residue mod p for the value of the cochain on each cell. Then we treat the cochain as an integral cochain. The conclusions of Theorem 2 remain valid except that (4) must be rewritten

$$(4') \quad \int_{B_r^{(p)}} \theta^r - \int_{\partial B_r^{(p)}} \Phi^{r-1} \equiv C_{(p)}^r \cdot B_r^{(p)} \pmod{p},$$

where $B_r^{(p)}$ is any r -chain (coefficients integers mod p) on M_n on which the integrals are defined.

When $C_{(p)}^r$ is a cocycle mod p , we cannot show that θ^r is regular over M_n (as was true for ordinary cocycles) for $\int_{\partial E_{r+1}} \theta^r \equiv 0 \pmod{p}$ but is not necessarily equal to zero. However, it is true that

$$\int_{Z_r^{(0)}} \theta^r \equiv 0 \pmod{p}$$

where $Z_r^{(0)}$ is a bounding integral cycle, for the values introduced by the singularities of θ^r are all multiples of p .

Since a cohomology class with integers mod p for coefficients is determined by its values on those integral chains which are cycles mod p , we state the following analog of Theorem 4.

THEOREM 5. *Let $C_{(p)}^r$ be a fixed r -dimensional cohomology class (coefficients integers mod p) on M_n and $Z_r^{(0)}$ and $Z_r^{(p)}$ arbitrary homology classes with integer and integer mod p coefficients respectively. Then there exists a closed r -form θ^r with singularities on $\partial^* C_{(p)}^{n-r}$ and an $r-1$ form Φ^{r-1} with singularities on $*C_{(p)}^{n-r}$ and with $d\Phi^{r-1} = \theta^r$ on $M_n - *C_{(p)}^{n-r}$ such that*

$$C_{(p)}^r \cdot Z_r^{(0)} \equiv \int_z \theta^r \pmod{p}, \quad C_{(p)}^r \cdot Z_r^{(p)} \equiv \int_z \theta^r - \int_{\partial z} \Phi^{r-1} \pmod{p},$$

where z is a cycle which represents the corresponding homology class and on which the integrals are defined.

6. Extension of de Rham's second theorem. It is known that any closed form is locally derived and (from the second theorem of de Rham) that a closed form whose periods are all zero is derived in the large. Further, any closed form determines a cohomology class (real coefficients). Let C^r represent this class and let $*C^{n-r}$ be the associated dual cycle. Then the following theorem is an extension of de Rham's theorem to the case where the periods are not all zero.

THEOREM 6. *Given any regular closed form ω^r on M_n , then there exists a form Ω^{r-1} regular except on $*C^{n-r}$ such that on $M_n - *C^{n-r}$*

$$\omega^r = d\Omega^{r-1}.$$

Let the forms corresponding to C^r be the θ^r and Φ^{r-1} of Theorem 3. Define $\lambda^r = \omega^r - \theta^r$. Then $d\lambda^r = 0$ and all the periods of λ^r are zero. Hence $\lambda^r = d\mu^{r-1}$ on M_n . Put $\Omega^{r-1} = \Phi^{r-1} + \mu^{r-1}$ on $M_n - *C^{n-r}$. Then

$$d\Omega^{r-1} = d\Phi^{r-1} + \lambda^r = \theta^r + \lambda^r = \omega^r.$$

7. Harmonic forms. The theorem of Hodge [2] states that among the numerous closed forms Θ^r which have the property described by de Rham's first theorem there is a unique harmonic form ω^r having this property. This suggests the following theorem.

THEOREM 7. *Theorems 2, 3, 4, and 5 remain valid if in their statements we replace " Θ^r " by "a harmonic form ω^r " and " Φ^{r-1} " by " Ω^{r-1} ". In Theorems 2, 3, and 4, ω^r is unique, but in Theorem 5 it depends upon the choice of residues mod p .*

PROOF. The method of orthogonal projection [3] states that in M_n (if Θ^r is regular everywhere) or in $M_n - \partial^*C^{n-r}$ (if Θ^r is singular on ∂^*C^{n-r}) there exists a unique, regular, harmonic form ω^r and a regular form μ^{r-1} such that

$$\Theta^r = \omega^r + d\mu^{r-1}.$$

If we put $\Omega^{r-1} = \Phi^{r-1} - \mu^{r-1}$, we have that

$$\int_{E_r} \omega^r - \int_{\partial E_r} \Omega^{r-1} = \int_{E_r} \Theta^r - \int_{\partial E_r} \Phi^{r-1} + \left\{ \int_{\partial E_r} \mu^{r-1} - \int_{E_r} d\mu^{r-1} \right\}.$$

Since the expression in braces on the right vanishes, the forms ω^r , Ω^{r-1} have the required properties. When ω^r are singular, they are called "harmonic forms of the third kind" as defined by Kodaira [3, Theorem 17].

We can now construct the correspondence between cohomology with real coefficients or with integer coefficients and harmonic forms as follows below. In doing so we need the notion of an *integral harmonic form*. Let ω^r be harmonic, and let Ω^{r-1} be given as in Theorem 6 such that on $M_n - *C^{n-r}$, $d\Omega^{r-1} = \omega^r$. Then ω^r is called *integral* if

$$\int_{E_r} \omega^r - \int_{\partial E_r} \Omega^{r-1}$$

is an integer for every cell E_r of K_r . With this understanding, we form the table of correspondences below:

<i>Cohomology</i>	<i>Unique harmonic forms ω^r and forms Ω^{r-1} such that $d\Omega^{r-1} = \omega^r$</i>
(1) Cochains (real coefficients)	(1) ω^r singular on ∂^*C^{n-r} Ω^{r-1} singular on $*C^{n-r}$
(2) Cochains (integer coefficients)	(2) As in (1) plus: ω^r is integral
(3) Cocycles (real or integer coefficients)	(3) As in (1) or (2) plus: ω^r regular on M_n
(4) Cobounding cocycles (real coefficients)	(4) As in (3) plus: $\omega^r = 0$; Ω^{r-1} unique harmonic
(5) Cobounding cocycles (integer coefficients)	(5) As in (4) plus: Ω^{r-1} integral
(6) Cobounding cocycles of order q —or torsion cocycles (integer coefficients)	(6) As in (4) plus: $q\Omega^{r-1}$ integral

A similar table can be readily constructed for cohomology with coefficients integers mod p .

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HAVERFORD COLLEGE,
HAVERFORD, PA., U. S. A.

TOPOLOGICAL GROUPS

SOME TOPOLOGICAL NOTIONS CONNECTED WITH A SET OF GENERATORS

P. A. SMITH

By a *local group*¹ we mean any system X which is like a group except that composition is not necessarily defined for all pairs of elements. The associative law takes the following form: if xy and yz are defined, then if one of the products $x(yz)$, $(xy)z$ is defined, so is the other and the two products are equal. It is assumed that each element of X has an inverse (in X) and that the products which are defined include $1x = x1 = x$, $xx^{-1} = 1$ for all $x \in X$. It is also assumed that $y^{-1}x^{-1}$ is defined whenever xy is. Evidently every group is a local group.

What is meant by a *local subgroup* of a local group Y is clear: X is a symmetric subset of Y which contains the identity and in which composition is simply the composition of Y cut down to X .

Which local groups can be imbedded in groups? It is easy to give examples of local groups which cannot be so imbedded. For example² let X consist of the distinct elements $1, x, y, z, u, v$ together with $a = ((x(yz))u)v$, $b = x(y(zu)v)$, the partial products of a and b , and the formal inverses of all these elements. Assume that no further compositions are defined except the trivial ones ($x1 = 1x = x$, $xx^{-1} = 1$, $\dots$) and those required by the associative law (e.g., $z(z^{-1}y^{-1}) = y^{-1}$). X cannot be a local subgroup of a group since $a \neq b$.

Let us call a mapping f of one local group into another a homomorphism if $f(x)f(y)$ is defined and equal to $f(xy)$ whenever xy is defined. We shall call a homomorphism *strong* if $f(x)f(y)$ is defined *only* when xy is defined. An isomorphism (one-one homomorphism) between local groups is not a symmetric relation but a strong isomorphism is, and two local groups which are strongly isomorphic to each other are identical in structure.

The question of imbedding can now be stated as follows: which local groups are strongly isomorphic to local subgroups of groups. One can of course bring additional elements of structure into the picture by considering topological local groups and local Lie groups (their definitions need not be repeated here). The following remark however shows that so far as imbedding is concerned, it is sufficient to consider the purely algebraic case. Let X be a local subgroup of a group Q generated by X . If X is topological and possesses a symmetric nucleus Y such that YY is defined (in X), then Q can be converted into a topological group in such a way that Y is a nucleus of Q . If X is a local Lie group, Q will be a Lie group.

The well-known theorem of Cartan concerning the existence of Lie groups with given structure constants implies that every local Lie group Y admits a

¹ Called by Malcev [C. R. (Doklady) Acad. Sci. URSS. vol. 32] a symmetric groupoid.

² See Malcev, loc. cit.

local subgroup X , nucleus of Y , which is imbeddable in a group, i.e., is strongly isomorphic to a local subgroup of a group. The existing proofs of the Cartan theorem make use of the infinitesimal elements of Lie groups and seem to depend, more or less indirectly, on the vanishing of certain 2-dimensional homology groups. We shall presently state a result (Theorem 5) about purely algebraic local groups which concerns the imbedding of homomorphisms of local groups rather than the local groups themselves. Again the vanishing of certain 2-dimensional connectivities is involved, this time quite explicitly.

We begin by defining certain groups $\mathfrak{p}_1, \mathfrak{p}_2$ associated with a local group X . We denote by $G(X)$ the group of equivalence classes of words formed with the elements of $X - \{1\}$, two words being equivalent if one is obtainable from the other by insertions and deletions of syllables of the form xx^{-1} . ($G(X)$ is not necessarily a free group since x may equal x^{-1} .) Let w be a word formed with the elements of $X - \{1\}$. Call *allowable* any insertion of parentheses into w which would reduce w to an element of X if all the indicated products were defined. Call w a *local* word if the products indicated by any allowable insertion of parentheses are all defined. If w is local, any insertion of parentheses will reduce w to an element $\mu(w)$ of X . Using the associative law it can be shown that $\mu(w)$ is independent of the choice of parentheses. Now let $V(X)$ be the totality of equivalence classes of local words w such that $\mu(w) = 1$. $V(X)$ is in general not a group. Let $D(X)$ be the smallest normal subgroup of $G(X)$ containing $V(X)$. Let

$$(1) \quad \mathfrak{p}_1(X) = G(X)/D(X).$$

Assume now that X is a local subgroup of a group Q generated by X . The mapping which associates to each word formed with the elements of $X - \{1\}$ its group product in Q induces a homomorphism α of $G(X)$ onto Q . Let $R(Q, X)$ be the kernel of α . Evidently $V(X) \subset R(Q, X)$, hence $D(X) \subset R(Q, X)$. Let

$$\mathfrak{p}_1(Q, X) = R(Q, X)/D(X).$$

We shall say that Q is *simply connected* relative to X if $\mathfrak{p}_1(Q, X) = \{1\}$.

THEOREM 1. *Let X be a local subgroup of a group Q generated by X and let Q' be a group. If Q is simply connected relative to X , any homomorphism of X into Q' can be extended in a unique manner to a homomorphism of Q' into Q .³*

Let $\tilde{Q}, Q$ be groups generated by local subgroups $\tilde{X}, X$ and let τ be a homomorphism of $\tilde{Q}$ onto Q . We shall call $(\tilde{Q}, \tilde{X}, \tau)$ an *even covering* of (Q, X) if $\tau|_{\tilde{X}}$ is a strong isomorphism of $\tilde{X}$ onto X .

THEOREM 2. *If X generates Q , there exists an even covering $(\tilde{Q}, \tilde{X}, \tau)$ of (Q, X) such that $\tilde{Q}$ is simply connected relative to X . This covering is essentially unique and $\tilde{Q} \cong \mathfrak{p}_1(X)$. (See (1).)*

³ Proofs of the principal theorems here stated will appear elsewhere.

Theorems 1 and 2 are easily proved counterparts of well-known theorems about topological groups.

Let $\chi = \{X\}$ be a class of local subgroups of a group Q and assume that each X generates Q . Call Q simply connected relative to χ if it is s.c. with respect to each X . Suppose that the class χ is determined by Q . Then simple connectedness relative to χ is a group theoretic invariant of Q . For example, χ might consist (1) of all X 's generating Q or (2) of all X 's generating Q and such that $p_2(X) = \{1\}$ (see below). In the first case it seems likely that the only groups which are s.c. relative to χ are the cyclic groups of orders 1, 2, 3.

A relation between algebraic and topological simple connectedness is given by the following theorem.

THEOREM 3. *Let Q be a locally connected topological group and let χ consist of all symmetric nuclei of Q . Then Q is simply connected in the topological sense⁴ if and only if Q is simply connected relative to χ .*

We turn to the definition of p_2 . Let X be a local group and let $A(X)$ be the totality of formal expressions.

$$a = (g_1 * v_1) \cdots (g_n * v_n)$$

where $g_i \in G(X)$, $v_i \in V(X)$. We introduce an associative multiplication into A by juxtaposition, and a neutral element a_0 . We define

$$a^{-1} = (g_1 * v_1^{-1}) \cdots (g_n * v_n^{-1}).$$

We allow $G(X)$ to operate on $A(X)$ according to the rule

$$g * a = (gg_1 * v_1) \cdots (gg_n * v_n).$$

Let β be the mapping $A \rightarrow D(X)$ defined by

$$\beta a = g_1 v_1 g_1^{-1} \cdots g_n v_n g_n^{-1}.$$

We introduce an equivalence $\sim$ into $A(X)$ by the relations

$$(2) \quad g * 1 \sim a_0$$

$$(3) \quad (g * v_1)(g * v_2) \sim g * v_1 v_2$$

$$(4) \quad gh * v \sim g * h v h^{-1}$$

$$(5) \quad a_1 a_2 a_1^{-1} \sim \beta a_1 * a_2.$$

These formulas of course define relations only when they are meaningful. Thus (2) is a relation only when $v_1, v_2, v_1 v_2$ are in $V(X)$; (3) is a relation only when v and $h v h^{-1}$ are in $V(X)$. In (4) on the other hand a_1, a_2 are arbitrary elements of $A(X)$.

The equivalence classes of $A(X)$ form a multiplicative system $\mathfrak{A}(X)$. It

⁴ As defined by Chevalley [*The theory of Lie groups*].

follows from (2) and (5) that $aa^{-1} \sim a_0$. Hence $\mathfrak{A}(X)$ is a group. It is easy to see that β induces a homomorphism

$$\gamma: \mathfrak{A}(X) \rightarrow D(X).$$

Let $\mathfrak{p}_2(X)$ be the kernel of γ . The group $\mathfrak{p}_2$ lies in the center of $\mathfrak{A}$ and is therefore abelian. For suppose a is a representative in A of an element of $\mathfrak{p}_2(X)$. Then $\beta a = 1$ so that $aa_1a^{-1} \sim 1 * a_1 = a_1$ for each $a_1 \in A(X)$.

Now assume that X is a local subgroup of a group Q generated by X . Let $K(Q, X)$ be the simplicial complex whose n -simplexes are subsets $(q_0, \dots, q_n)$ of Q of cardinal number $n + 1$ such that $q_i^{-1}q_j \in X$ ($i, j = 0, \dots, n$). It is easy to see that K is connected. It can be shown that

$$\mathfrak{p}_1(Q, X) = \pi_1(K), \quad \mathfrak{p}_2(X) = \pi_2(K)$$

where $K = K(Q, X)$ and π denotes homotopy groups.⁵

If Q is a topological group satisfying certain conditions, $\mathfrak{p}_1$ and $\mathfrak{p}_2$ determine the first and second homotopy groups of Q . As concerns π_1 , the situation is partially described by Theorem 3. We pass to the case of π_2 . Let X be a local subgroup of a local group Y . The relation $X \subset Y$ gives rise in a natural manner to a homomorphism $i: \mathfrak{p}_2(X) \rightarrow \mathfrak{p}_2(Y)$. We denote the image $i\mathfrak{p}_2(X)$ by $\mathfrak{p}_2(X, Y)$. Now let Q be a topological group and consider the following conditions:

- I. For every nucleus V there exists a nucleus $U \subset V$ such that singular 1-spheres in U are nullhomotopic in V .
- II. There exists a nucleus W such that singular 2-spheres in W are nullhomotopic (in Q).

THEOREM 4. *Let Q be an arcwise connected topological group satisfying conditions I and II. Then there exists a nucleus Y_0 such that every symmetric nucleus X contained in Y_0 admits a symmetric subnucleus X such that $\mathfrak{p}_2(X, Y) = \pi_2(Q)$.*

It is known that if Q is a Lie group, then $\pi_2(Q)$ is trivial. This is not true however for topological groups in general.⁶ We shall see that the problem of determining whether or not the second homotopy group of a topological group Q is trivial can be reduced to a purely algebraic study of the "extensions" of the local subgroups of Q . How the algebraic structure of Q determines the group $\pi_2(Q)$ when it is not trivial is not yet known.

Let Y be a local group. An *extension* of Y is a pair (E, ϕ) where E is a local group and ϕ a strong homomorphism of E onto Y . Note that if Y is a group, so is E . Let (E, ϕ) be an extension of Y and let X be a local subgroup of Y . It is

⁵ Since K is not a space, π_1 and π_2 must be understood as being defined combinatorially. Thus π_2 is the second homology group, based on finite chains and integer coefficients, of the universal covering of K .

⁶ An example of an arcwise topological group Q for which $\pi_2(Q)$ is infinite cyclic was communicated to me independently by Dr. H. C. Wang and Dr. Robert Taylor.

easy to see that by taking $E' = \phi^{-1}X$ and $\phi' = \phi|E'$, we obtain an extension (E', ϕ') of X . We write $(E', \phi') = (E, \phi)|X$.

Let Y, Z be local groups such that $Y \subset Z$. An extension (E, ϕ) of Y will be called *extendible over Z* if there exists an extension (E_1, ϕ_1) of Z such that $(E, \phi) = (E_1, \phi_1)|Y$. We shall say that (E, ϕ) is extendible over Z from X , where X is a local subgroup of Y , if $(E, \phi)|X$ is extendible over Z . We can now state the following theorem.

THEOREM 5. *Let X and Y be local subgroups of a group Q with X a local subgroup of Y . Assume that X generates Q , that Q is simply connected with respect to X , and that Y contains no elements of order 2. Then a necessary and sufficient condition that every extension of Y be extendible over Q from X is that $\mathfrak{p}_2(X, Y) = \{1\}$.*

Suppose Q is a simply connected topological group satisfying conditions I and II and that Q does not possess "arbitrarily small" elements of order 2. From Theorems 4 and 5 we see that there exists symmetric nuclei X and Y in Q with $X \subset Y$ such that the question whether or not $\pi_2(Q)$ is trivial is equivalent to the question whether or not all extensions of Y are extendible over Q from X .

By using Theorems 1 and 2 it is easy to see that an extension (E, ϕ) of a subgroup X of a group Q generated by X is extendible over Q if and only if E is imbeddable in a group. This suggests another form for Theorem 5; we state it only for the case $X = Y$.

THEOREM 5'. *Let X be a local subgroup of a group Q generated by X and assume that Q is simply connected relative to X and that X contains no elements of order 2. A necessary and sufficient condition that every local group E which can be mapped onto Q by a strong homomorphism be imbeddable in a group is that $\mathfrak{p}_2X = \{1\}$.*

We now state in rough form an application of Theorem 5' to the problem of imbedding local groups in groups. Let Z be a local group and N a "normal" local subgroup of Z . Assume that the local groups N and $Y = Z/N$ are imbeddable in groups, that Y contains no elements of order 2, and that $\mathfrak{p}_2(Y) = \{1\}$. Then Z is imbeddable in a group.

In fact, it can be shown in a fairly straightforward manner that the "cosets" of N are contained in larger sets whose union E is a local group which contains Z and is mappable onto $Y = Z/N$ by a strong homomorphism (determined by the natural mapping $Z \rightarrow Z/N$). From Theorem 5', E is imbeddable in a group, and therefore so is Z .

An accurate statement of this theorem is complicated by the fact that the concepts "normal local subgroup" and "local quotient group" are somewhat nebulous. When accurately stated, the theorem asserts that a certain local subgroup Z' of Z , rather than Z itself, is imbeddable. When Z is topological, Z' can be taken to be a nucleus. The theorem can be given a formulation which

involves $\mathfrak{p}_2(X, Y)$ rather than $\mathfrak{p}_2(X)$. If the group Q in which Z/N is assumed to be imbeddable is topological and satisfies conditions I and II, the condition $\mathfrak{p}_2(X, Y) = \{1\}$ can be replaced by $\pi_2(Q) = \{1\}$ (see Theorem 4). It is not hard to see that in this form, the theorem furnishes a new proof of the fact that every local Lie group Z has a nucleus which is imbeddable in a group, provided one makes use of the proposition that the second homotopy group of every semi-simple group is trivial. One has only to take N to be a maximal solvable local subgroup of Z , so that N/Z is semi-simple.⁷

COLUMBIA UNIVERSITY,
NEW YORK, N. Y., U. S. A.

⁷ That solvable and semi-simple local Lie groups are imbeddable in groups (or at least admit imbeddable symmetric nuclei) is little more than a triviality.

PROPERTIES OF FINITE-DIMENSIONAL GROUPS

DEANE MONTGOMERY

Topological groups and topological transformation groups are subjects without a very clear dividing line. To mention one connection, a subgroup H of a group G can be regarded as a transformation group acting on G by either left or right translations or by inner automorphisms. This paper recalls a few of the results and problems in both topics.

Let G be a topological group and M a topological space. A function $f(g; x) = g(x)$ defined and continuous on $G \times M$ with values in M is called a transformation group if

- (a) for each fixed g , $g(x)$ is a homeomorphism of M onto itself,
- (b)
$$g_1[g_2(x)] = (g_1g_2)(x).$$

The transformation group is called effective if only the identity leaves all of M fixed. The set $G(x)$ for any x in M is called the orbit of x . The spaces G and M will always be locally compact and separable metric so that dimension theory may be used. When G and M coincide and G acts on itself by translation, the subject becomes the study of the topological group G .

A group which has a simple structure may offer difficult questions when operating as a transformation group. For example, the ways in which a cyclic group of order 2 can operate on a manifold, even on E^3 , are far from completely known.

If h is any homeomorphism of M onto itself, then the above transformation group determines another in which $g(x)$ is replaced by $(hgh^{-1})(x)$. Two transformation groups related in this way are called equivalent and it is natural to try to determine conditions which imply the equivalence of two transformation groups. When equivalence is likely but unproved, it can be asked if some of the homology or other properties of the two transformation groups are the same.

It is often conjectured that many transformation groups are equivalent to known or comparatively simple ones. One question of this kind is the problem of Hilbert which asks, when G and M are both manifolds, whether coordinates may be so introduced in G and M that $f(g; x)$ becomes analytic in both variables. This is probably true in 1 below and at least in part in 2.

1. G is a manifold which acts on itself by group translation.
2. G is a compact manifold, and M is any manifold. In this case it follows from the work of von Neumann [13] that G is a Lie group, so that analytic parameters can be introduced into G ; thus the problem 2 is that of also choosing x so that $f(g; x)$ is simultaneously analytic.

Case 2 suggests the following:

2'. If G is compact (not necessarily a manifold) and acts effectively on a manifold M , does this imply that G is a manifold, and hence a Lie group?

In connection with 2 and 2', Zippin and the author have shown [10; 8; 16] that any compact connected group acting effectively on a three-dimensional manifold M must be a Lie group. If $M = E^3$, we showed further that G must be equivalent either to the group of all rigid motions about an axis or to the group of all rigid motions about a point. The case where G is a compact, zero-dimensional, effective group on three and higher dimensional manifolds remains open. Presumably such a G must be finite as suggested in 2', but this has not been proved, and even if this is assumed and $M = E^3$, it is not known whether G is equivalent to a group of orthogonal transformations. This latter question has not been answered even in the differentiable case. It has been shown that a compact effective zero-dimensional group acting on a two-dimensional manifold is finite.

When M is an n -dimensional manifold and G is compact, effective, and has locally connected orbits, Zippin and the author have shown that G must be a Lie group [12]. This result is related to the fact that a pointwise periodic homeomorphism of a manifold must be periodic, that is, if every point has a period, then these periods are bounded [7]. If M is differentiable, G is compact and effective, and each element of G is of class C^1 , then G is a Lie group and $f(g; x)$ is of class C^1 in both variables. Bochner and the author have shown [1] that if G is a Lie group and if $f(g; x)$ has certain properties of differentiability or analyticity with respect to x , then $f(g; x)$ has the same properties with respect to g . If M is a manifold and G is finite, many beautiful results have been obtained by Newman and Smith.

Even when G is a Lie group and M is a differentiable or analytic manifold and if $f(g; x)$ is assumed simultaneously differentiable or analytic, then many problems remain. At first glance this might seem unlikely after the great work of Lie and his followers. But much of their work is concerned with local questions and is not concerned with the topology of orbits or the interconnection of the topology of orbits, group, and space. Much progress has been made recently, especially by the French mathematicians, on the important case where a compact Lie group acts transitively on a compact manifold.

When a compact group acts intransitively, the orbits decompose the space in a manner reminiscent of a fibering, but there may be various kinds of singularities, and this suggests that it might be useful to consider fiberings with singularities to a greater extent than has been done so far.

For the remainder of the paper the topological group G will be considered for its own sake. As has been remarked this is a special case of the study of transformation groups. It is helpful to assume that G is n -dimensional and this will be done from this point on. Dimension is used in the sense of set theory and does not necessarily mean that G has local coordinates.

If H is a closed subgroup, it can be shown, using a theorem of Hurewicz, that $\dim G \leq \dim H + \dim G/H$.

It is probable that

$$\dim G = \dim H + \dim G/H,$$

but this has been proved only in a few special cases. It has not been proved in general that $\dim G/H$ is finite, although the author has proved this when H is abelian. If H is n -dimensional, and G is connected, then $H = G$.

In low dimensions the structure of G can be analyzed completely, although not without considerable effort. If $\dim G = 1$, G connected, then G is either compact or isomorphic to the group of real numbers. If $\dim G = 2$, G connected, then G is either abelian and has a known structure or else it is the nonabelian group of transformations of the form $ax + b$, a positive. If G is locally connected and $\dim G = 1, 2$, or 3 , then G must be locally euclidean. In these cases, that is, G locally connected and $\dim G = 1, 2, 3$, it is known that G is a Lie group [4].

For a general n , the local topological structure of G is known when G is compact or abelian by the work of von Neumann and Pontrjagin [1]. In this case there is an open set U including e with U a direct topological and group theoretical product

$$U = Z \times C$$

where Z is a compact zero-dimensional group and C is an n cell and a local Lie group. In the general case not nearly this much is known, but, using Gleason's result on the existence of an arc in a locally compact connected group [2], it has been shown that there is a neighborhood U of e with U a topological product

$$U = Z \times C$$

where Z is a compact zero-dimensional set and C is a connected, locally connected, invariant, n -dimensional local group [5]. By retopologizing the subgroup generated by C it is possible to prove the following theorem.

THEOREM 1. *Let G be a locally compact connected n -dimensional group. Then there exists a connected locally connected locally compact n -dimensional group L and a continuous one to one homomorphism of L into G such that $h(L)$ is everywhere dense in G .*

A familiar illustration occurs when G is a solenoid and L is the group of real numbers. This result reduces many questions about groups to the locally connected case.

Locally connected n -dimensional groups have been shown to have a few properties which make them resemble n -dimensional manifolds [6]. If such a group contains a closed n -dimensional subset E , then E contains an inner point. A similar fact is true for an n -dimensional homogeneous space of a locally connected group. In either case a sufficiently small compact set which carries an

essential $n - 1$ cycle must separate, and a closed $(n - 1)$ -dimensional set must separate locally. Local homology connectedness can be proved in some dimensions. More generally, if a space M has a certain property called local homogeneity which, speaking roughly, requires that the space can be deformed, locally, rather freely, then M has the above-mentioned properties of a manifold. The property of local homogeneity is possessed by all manifolds but in how far it may characterize manifolds is not known and is probably worth study.

In another direction and by using some of the above results, Zippin and the author [9] have shown the following:

THEOREM 2. *Let G be a locally compact connected n -dimensional group, $n > 0$, which is not compact. Then G contains a closed subgroup isomorphic to the real numbers.*

It was shown previously by Gleason [3] that every n -dimensional group, $n > 0$, contained a one-parameter subgroup and by Zippin [15] that every two-ended group, whether finite-dimensional or not, contained a subgroup isomorphic to the real numbers.

One of the tools used in proving Theorem 1 is the device of considering G as acting on itself by inner automorphisms. A brief outline of the proof is now given. Assuming the theorem false implies the existence of a compact connected abelian subgroup of positive dimension. Using inner automorphisms shows then that every element of G is in such a compact abelian subgroup and this turns out to be a contradiction to the noncompactness of G .

Theorem 1 has as a corollary that G is a topological product

$$G = F \times R$$

where R is homeomorphic to a line. This follows from the fact that there is a cross section in the large of the cosets of a subgroup isomorphic to the reals. It is probable that a two-ended group is the direct product of a compact group and the group of real numbers, but this has not been shown.

Again by considering the group as acting on itself by inner automorphisms, Zippin and the author [11] have proved the following theorem.

THEOREM 3. *Let G be a locally compact, connected, n -dimensional group, $n > 1$, which is not compact. Then G contains a closed connected two-dimensional subgroup which is not compact.*

The corresponding fact is not true for compact groups since the proper orthogonal group in three variables and its covering group do not contain two-dimensional subgroups, but these two groups are the only exceptions to the statement that every locally compact n -dimensional group, $n > 1$, contains a closed two-dimensional subgroup.

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INSTITUTE FOR ADVANCED STUDY,
PRINCETON, N. J., U. S. A.

SOME PROPERTIES OF (L)-GROUPS¹

KENKICHI IWASAWA

One of the main problems in the theory of topological groups at the present time is the study of the structure of locally compact groups in both algebraical and topological aspects. In some particular cases, for example, in the case of Lie groups, we already know much about them. There are indeed still many difficult problems left unsolved concerning Lie groups, but we are in possession of powerful methods from algebra and analysis by which we can research deeply in these groups. The situation is similar for compact groups or locally compact abelian groups, for they are projective limits of Lie groups and we can study their structure by approximating them as precisely as we want by Lie groups. Thus the fifth problem of Hilbert is solved for these groups.

However if we leave these particular cases and consider locally compact groups in general, our knowledge about them is, it may be said, as yet insufficient. Indeed we have no general method by which we can analyze the structure of these groups as we do in the case of compact or abelian groups. Therefore it seems advisable to me to consider first the class of those locally compact groups which can be studied by known methods just as in the special cases mentioned above, and then ask the relation between these groups and general locally compact groups. Such are the class of (L)-groups, namely the class of those locally compact groups which are projective limits of Lie groups.²

As one expects from this definition, (L)-groups, in particular connected (L)-groups, have properties similar to those of Lie groups in various aspects. First, it is easy to see that subgroups and factor groups of (L)-groups are also (L)-groups.³ Moreover we can prove the following theorem.

THEOREM 1.⁴ *Let G be a connected locally compact group and N a closed normal subgroup of G . If N and G/N are both (L)-groups, then G itself is an (L)-group.*

The proof is essentially based upon the following lemma.

¹ This address was listed on the printed program under the title *Locally compact groups*.

² In my paper, *On some types of topological groups*, Ann. of Math. vol. 50 (1949), I defined the (L)-group as a locally compact group G containing a system of normal subgroups N_α such that G/N_α are Lie groups and such that the intersection of all N_α is e . If G is a projective limit of Lie groups, G surely contains such a system of normal subgroups, and the converse can also be proved to be true when we assume that G is connected. But we do not know whether these two definitions are completely equivalent.

³ If we take the former definition of (L)-groups, we cannot yet decide whether a factor group of an (L)-group is always an (L)-group or not.

⁴ For the proof of following theorems see the paper of K. Iwasawa, *On some types of topological groups*, Ann. of Math. vol. 50 (1949). Cf. also A. M. Gleason, *On the structure of locally compact groups*, Proc. Nat. Acad. Sci. U. S. A. vol. 35 (1949).

LEMMA. *Let G and N be as above. If N and G/N are both Lie groups, G itself is also a Lie group.*

Theorem 1 and the foregoing remark show us that the class of connected (L)-groups is closed under the group-theoretical operations, such as taking subgroups, forming factor groups or group extensions. Now Lie groups are of course (L)-groups, and compact groups and locally compact abelian groups are also known to be (L)-groups. Therefore those connected locally compact groups, which can be obtained from these "known groups" by successive group-theoretical operations as stated above, are always (L)-groups. In particular, connected locally compact solvable groups are (L)-groups.

A connected (L)-group has a characteristic local structure. We have namely the following theorem.

THEOREM 2. *A connected locally compact group G is an (L)-group if and only if G is locally the direct product of a local Lie group L and a compact normal subgroup K .⁵*

This theorem implies in particular that a locally Euclidean (L)-group is a Lie group. Therefore, as every connected locally compact solvable group is an (L)-group, every locally Euclidean solvable group is a solvable Lie group.⁶

As to the global structure of (L)-groups we have the following theorem.

THEOREM 3. *Let G be a connected (L)-group. Any compact subgroup of G is then contained in some maximal compact subgroup of G , whereas such maximal compact subgroups of G are all connected and conjugate to each other. Let K be one of them. G contains then subgroups $H_1, \dots, H_r$, which are all isomorphic to the additive group of real numbers and are such that any element g in G can be decomposed uniquely and continuously in the form*

$$g = h_1 \cdots h_r k, \quad h_i \in H_i, k \in K.$$

In particular, the space of G is the cartesian product of the compact space of K and that of $H_1 \cdots H_r$, which is homeomorphic to the r -dimensional Euclidean space.

This theorem is obviously a generalization of Cartan-Malcev's theorem on the topological structure of Lie groups.

By Theorem 3 we see that the topological structure of a connected (L)-group G is completely determined by that of its maximal compact subgroup K . For example, the study of the topological structure of a connected locally compact solvable group G is reduced to that of a compact abelian group, for we can prove in this case that the maximal compact subgroup K is an abelian group.

⁵ For a simple proof cf. A. Borel, *Limites projectives de groupes de Lie*, C. R. Acad. Sci. Paris vol. 230 (1950).

⁶ Chevalley's theorem.

Now, by making use of Theorem 3, we can also prove the following theorem.

THEOREM 4. *A connected locally compact group G contains a uniquely determined maximal connected solvable normal subgroup R , in which any other connected solvable normal subgroups of G are contained.*

We call such an R , just as in the case of Lie groups, the *radical* of G . We also say that G is semi-simple if the radical R of G is equal to e . In general a connected locally compact group G is an extension of the solvable radical R by the semi-simple group $\bar{G} = G/R$.

For (L)-groups we have moreover the following generalization of E. E. Levi's theorem.

THEOREM 5. *Let G be a connected (L)-group and R the radical of G . Then G contains a subgroup S , such that*

- (i) *S is the homomorphic image of a connected semi-simple (L)-group S' in G ,*
- (ii) *$RS = G$ and $R \cap S$ is a totally disconnected normal subgroup of G .⁷*

Now it can be proved that a connected locally compact group G contains not only the radical R , but also a uniquely determined maximal connected compact normal subgroup N . By making use of these facts we have the following theorem.

THEOREM 6. *A connected locally compact group G contains a uniquely determined maximal normal subgroup Q of type (L), and G/Q contains no normal subgroup of type (L) other than e .*

On the other hand, it follows immediately from the definition of (L)-groups that a connected locally compact group G contains a uniquely determined minimal normal subgroup Q' of type (L). These facts show us some relations between the class of all connected (L)-groups and general connected locally compact groups. But we have as yet no essential result concerning the situation of connected (L)-groups in the set of all connected locally compact groups. However we have the following conjecture:⁸

(C₁) Any connected locally compact group is an (L)-group.

By theorem 6 this conjecture is easily seen to be equivalent to the following:

(C₂) A connected locally compact group which contains no arbitrary small normal subgroup is a Lie group.

⁷ Cf. A. Borel, *Limites projectives de groupes de Lie*, C. R. Acad. Sci. Paris vol. 230 (1950) and Y. Matsushima, *On the decomposition of an (L)-group*, Journal of the Mathematical Society of Japan vol. 1 (1950).

⁸ Cf. K. Iwasawa, *On some types of topological groups*, Ann. of Math. vol. 50 (1949) and A. M. Gleason, *On the structure of locally compact groups*, Proc. Nat. Acad. Sci. U. S. A. vol. 35 (1949).

We can also prove that (C_1) , (C_2) are equivalent to the following:

(C_3) A connected locally compact simple group is a simple Lie group.

If these conjectures could be proved to be true, we would have not only a complete solution of the fifth problem of Hilbert, but also decisive progress in the theory of locally compact groups.

INSTITUTE FOR ADVANCED STUDY,
PRINCETON, N. J., U. S. A.

ONE-PARAMETER SUBGROUPS AND HILBERT'S FIFTH PROBLEM

A. M. GLEASON

The affirmative solution of Hilbert's fifth problem requires that we bridge the gap between topologico-algebraic structure and analytic structure. In building this bridge we quite naturally seek an intermediate island on which to rest the piers. Such an island is provided by the one-parameter subgroups. One-parameter subgroups are themselves a topologico-algebraic concept and their existence can be demonstrated, in some cases at least, by the methods of topological algebra. On the other hand the one-parameter subgroups are perhaps the most striking feature of a Lie group and it is known that the analytic structure can be recovered from them.

A one-parameter subgroup of a group G is a subgroup which is a (continuous) homomorphic image of the additive group of real numbers R . We do not require that the subgroup be closed. The structure of such a subgroup can be quite complicated, even if the group G is locally compact, a condition which we shall assume throughout. If we consider only a part of the subgroup corresponding to a small segment of the reals including 0, the complications vanish and all such *local* one-parameter subgroups look the same. In a Lie group there is a neighborhood U of the identity e such that every element x of U is on a one-parameter subgroup. Furthermore this one-parameter subgroup is unique if we require that it go directly from e to x without leaving U . We shall say that a group has a canonical family of one-parameter subgroups if there exists a neighborhood U with these properties.

Our intermediate goal is to prove that every locally Euclidean group has a canonical family of one-parameter subgroups. Quite recently great strides have been made toward this objective.

An important class of groups is the class of those which contain no small subgroups; that is, those which have a neighborhood of the identity containing no entire subgroup except (e) . It has long been known that a group with no small subgroups contains a one-parameter subgroup. An extension of this result was made by Chevalley and the author independently: A locally connected group of finite dimension which contains no small subgroups has a canonical family of one-parameter subgroups. Unfortunately little is known about the existence or nonexistence of small subgroups in locally connected groups of finite dimension. However, under the stronger hypothesis that the group be locally Euclidean, Newman has shown that there is a neighborhood of the identity containing no finite subgroup. Smith has extended his investigation and shown that if there are arbitrarily small subgroups H in a locally Euclidean group, then some of them must satisfy the implausible relation $\dim G/H > \dim G$.

Without the hypothesis concerning small subgroups we can say less, but still a great deal. The author proved that every n -dimensional group G ($n > 0$)

contains a one-parameter subgroup. Montgomery and Zippin have shown that, provided G is not compact, this subgroup can be chosen isomorphic to R . The method of proof, in both cases, is to prove the existence of subgroups of lower dimension, eventually winding up with a subgroup of dimension one. This method does not seem applicable to proving that there is a canonical family of one-parameter subgroups.

On the other side of our island the situation is not so bright. This is to be expected; it is here that we must make the transition from topological algebra to analysis. On the analytical side of the channel, the stringency of the conditions leading to analytic structure have gradually been relaxed from requiring three times differentiable coordinates to certain rather strong Lipschitz conditions; but all conditions have been truly analytic in character, and it seems safe to say that the first purely analytic result derived by the methods of topological algebra will prove decisive.

Consider the class Φ of homomorphisms of R into G . (There will of course be many distinct homomorphisms onto each one-parameter subgroup.) If G is a Lie group, every homomorphism of R into G has the form $t \rightarrow \exp tX$ where X is an element of the Lie algebra $\mathfrak{g}$ of G ; hence, we may identify Φ with $\mathfrak{g}$. Evidently we should attempt to introduce the structure of a Lie algebra into Φ .

To introduce the additive structure into Φ , we again turn to the theory of Lie groups for our cue. We have the basic formula

$$(1) \quad \exp(X + Y) = \lim_{n \rightarrow \infty} (\exp X/n \exp Y/n)^n.$$

To carry this over to Φ we must prove that, for any two homomorphisms φ_1 and φ_2 of R into a locally Euclidean group, $\lim_{n \rightarrow \infty} (\varphi_1(1/n)\varphi_2(1/n))^n$ exists. If this is true, we can define addition in Φ quite simply by

$$(2) \quad (\varphi_1 + \varphi_2)(t) = \lim_{n \rightarrow \infty} (\varphi_1(t/n)\varphi_2(t/n))^n.$$

The commutativity of this operation is easily proved, but the associativity is in doubt. Scalar multiplication is defined, of course, by $(\alpha\varphi)(t) = \varphi(\alpha t)$, and it satisfies the distributive law with respect to addition.

If these ideas can be carried out, making Φ a linear vector space, it will follow that G is a Lie group. For if C is the center of G , then G/C will be represented faithfully by the linear transformations induced on Φ by the inner automorphisms of G . Hence G is a generalized Lie group, and, being locally Euclidean, it is a Lie group.

It may be noted that the program here outlined does not actually require that we have a canonical family of one-parameter subgroups. It is sufficient that the set of one-parameter subgroups generate G , and this can be shown to be no essential restriction.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS., U. S. A.

RECENT DEVELOPMENT OF KNOT THEORY AT PRINCETON

R. H. Fox

I shall report¹ here on some of the principal results in knot theory obtained during the past five years by myself and my colleagues, and try to indicate the trend of these developments.²

Let us examine the question: What is knot theory and what is its proper domain? The subject matter of knot theory is usually considered to be the situation in euclidean 3-space of simple closed polygons or of systems of μ disjoint simple closed polygons, $1 \leq \mu < \infty$. This description of what I may call classical knot theory tends, by its narrowness, to isolate the subject from the rest of topology. It is to be hoped that the various special theorems which make up classical knot theory will eventually turn out to be particular cases of general topological theorems. In working toward this end the following principles seem almost obvious:

(A) *The class of polygons should be replaced by a suitable topologically defined class of curves.* The minimal topologically invariant class is the class of *tame curves*,³—those which can be transformed into polygons by autohomeomorphisms of the underlying manifold. However an effective topological definition of this class is lacking. The maximal class, the class of *all simple closed curves*, has the disadvantage that it introduces into the subject the new complication of local pathology.⁴ It would seem to be preferable to consider the intermediate class of *smooth curves*,—a curve is smooth if it can be transformed into a polygon by a homeomorphism of one of its neighborhoods onto a regular neighborhood of the polygon. It is natural to conjecture that every smooth curve is tame, but this has not been proved.

(B) *Euclidean 3-space should be replaced by other compact⁵ 3-manifolds.* Since the complement of a neighborhood of a smooth curve is a manifold with boundary, it would seem advantageous to admit these as well as the closed manifolds.

In accordance with these principles we see that *the natural domain of knot theory is the study of the types of ordered systems $K = (K_1, K_2, \dots, K_\mu)$ of disjoint oriented smooth simple closed curves in the interior of a compact 3-manifold*

¹ In the interest of brevity several of the definitions are intentionally vague or even inaccurate.

² Since I am speaking about the work of the group with whom I have been associated at Princeton, I shall not be discussing the various recent results obtained in this field by Reidemeister and Seifert and those associated with them at Marburg and Heidelberg, although there has been a certain amount of overlapping. References to their results may be found in [24] and [20]. Numbers in brackets refer to the references at the end of the paper.

³ Terminology introduced in [3].

⁴ Instructive examples of such pathology have been studied by E. Artin and R. Fox [3], R. Fox [10], W. A. Blankinship [6], and W. A. Blankinship and R. H. Fox [7].

⁵ In classical knot theory euclidean 3-space may be replaced with advantage by spherical 3-space S , which is a compact manifold.

M . (Such a system may be called a *knot* of $\mu \geq 1$ components.) Two knots belong to the same *type* if there is an isotopy of M on itself which transforms the one knot into the other.

Knot theory really began in 1910 with the introduction [9] of *the group of the knot*. This is the fundamental group of the complement, $G = \pi(M - K) = \pi(M - V)$, where V is the union of sufficiently small open tubular neighborhoods V_i of the component knots K_i , $i = 1, \dots, \mu$. It is known that this invariant is insufficient; examples to this effect have been given by Dehn,⁶ Seifert,⁶ and J. H. C. Whitehead.⁶ However it is easy to replace the group by a more comprehensive invariant, *the group system of the knot*. This consists of the groups $\pi(\dot{V}_i)$ and $\pi(\bar{V}_i)$ together with a preferred generator for the infinite cyclic group $\pi(\bar{V}_i)$, $i = 1, \dots, \mu$, the groups $\pi(M - V)$ and $\pi(M)$, and the indigenous homomorphisms. These latter may be represented by μ diagrams

$$\begin{array}{ccc} \pi(\dot{V}_i) & \longrightarrow & \pi(\bar{V}_i) \\ \downarrow & & \downarrow \\ \pi(M - V) & \longrightarrow & \pi(M) \end{array} \quad i = 1, \dots, \mu,$$

and there is the usual commutativity around the diagrams. Every known example of insufficiency of the group is resolved by the group system. For Dehn's example this was shown by Dehn;⁶ for Whitehead's example it is quite trivial; for Seifert's example I have been able to accomplish this⁷ by considering the representations of G in the symmetric group of order 5. Probably the known invariants of knot theory can all be expressed as invariants of the group system. The question of how strong an invariant the system really is may possibly depend on a resolution of the present unsatisfactory state of Dehn's lemma.

In 1928 Alexander [1] associated with each polygonal knot in 3-space an equivalence class of matrices $\{\mathbf{M}\}$; invariants of this class and hence of the given knot are the elementary divisors and in particular their product Δ . Alexander also showed how to derive an $\mathbf{M}$ from the group system; his algorithm requires one first to find a presentation of the group of a special kind called canonical. I have succeeded in finding an algorithm which is applicable to any presentation in which the number of generators is finite. This algorithm makes use of the free differential calculus,⁸ and in fact it was the search for such an algorithm which led me to invent this calculus. With this "invariant" algorithm the matrices $\{\mathbf{M}\}$, their elementary divisors, and the Alexander polynomial Δ are defined for an arbitrary finitely generated group G and hence, in particular, for any knot in the interior of any compact 3-manifold. Δ is an element of the integral ring RB of the betti group B of G ; it depends on a choice of basis for the free abelian group B and is determined up to a factor ϵ which is a unit of

⁶ M. Dehn [8], H. Seifert [23], J. H. C. Whitehead [27]. In the examples of Dehn and Seifert, $\mu = 1$. In the example of Whitehead, $\mu = 2$, but his example has the additional feature that the complements not only have the same fundamental group but are even homeomorphic.

⁷ Unpublished.

⁸ [11] and [12]. A full account has not yet been published.

the ring RB . Because of the indeterminacy, Δ is essentially a polynomial in ν variables, where ν is the rank of the free abelian group B . For the case of a knot in 3-space S , $\nu = \mu$ and a basis $t_1, \dots, t_\mu$ for B is uniquely determined by the group system.

Alexander conjectured and Seifert proved [22] that the polynomial of a tame connected knot in S is always symmetric: $\Delta(1/t) = t^\lambda \Delta(t)$. G. Torres [25] has recently proved that the polynomial of a tame knot of any multiplicity $\mu > 1$ is symmetric: $\Delta(1/t_1, \dots, 1/t_\mu) = (-1)^{\mu\lambda_1} \dots t_\mu^{\lambda_\mu} \Delta(t_1, \dots, t_\mu)$. Whether symmetry is a property of Δ for knots in other 3-manifolds than S is not at present known; it is definitely not a property of Δ for an arbitrary finitely generated group.

The trivial property $\Delta(1) = 1$ of the Alexander polynomial of a tame connected knot in S has been generalized in several directions. For tame knots in S with $\mu = 2$, Torres⁹ has shown that $\Delta(t_1, 1) = ((t_1^l - 1)/(t_1 - 1)) \Delta(t_1)$, where $\Delta(t_1)$ is the polynomial of the first component knot and l is the linking number. Corollary: $\Delta(1, 1) = l$.

The property $\Delta(1) = 1$ has also been generalized¹⁰ to the case of a tame connected knot in the interior of a compact 3-manifold M . Let N denote the kernel of the injection homomorphism $G \rightarrow \pi(M)$. I have shown that the group $G/[N]$ is an invariant of the injection $\pi(\bar{V}) \rightarrow \pi(M)$. (This is a refinement of that part of the Alexander duality theorem which states that $H = G/[G]$ is an invariant of this injection.) From a slight but important strengthening of this theorem it follows that the residue class¹ $\bar{\Delta}$ of Δ modulo the kernel ideal of the injection homomorphism $RH(M - K) \rightarrow RH(M)$ is also an invariant of this injection (at least for polygonal K). Thus, given M , there is invariantly associated to each element of $\pi(M)$ a residue class $\bar{\Delta}$. This invariant of M is sufficiently strong to duplicate Reidemeister's [18] combinatorial classification of the lens spaces, and there is some hope that, by this new method, the classification can be made topological.

The covering spaces of spherical 3-space S branched over a tame knot K are invariants of the knot, as was pointed out by Alexander [26, p. 158] in 1920. Alexander [2], Reidemeister [19], and Seifert [21], [22] gave useful algorithms for calculating the homology groups and linking invariants for the cyclic coverings S_g , $g = 1, 2, \dots$, branched over a connected tame knot. It was not clear to me whether the results were topologically invariant since branched covering spaces had been defined only combinatorially. I have resolved¹¹ this difficulty as follows:¹ A pair (X, f) consisting of a space X and a continuous mapping f of X into a space Y is called a *covering* of $f(X)$ if the collection consisting of the components of the inverse images of the open sets of Y is a basis for the open sets of X . It may be shown that every covering (X, f) of $f(X)$ can be uniquely

⁹ [25]. A similar result is proved for $\mu > 2$, and several other theorems of Seifert are generalized to the cases $\mu > 1$.

¹⁰ Unpublished. This solves a problem that I proposed at the Princeton Bicentennial Conference. See Ann. of Math. vol. 50 (1949) p. 247.

¹¹ Unpublished.

extended to an "optimal" covering (X^*, f^*) of $\overline{f(X)}$. If we apply this to an "unbranched covering" of $M - K$ with $Y = M$, there results a unique "branched (over K) covering" of M which can be recognized intrinsically as a branched covering.¹² It is of interest to note that application to the case of a locally compact space X and the identity mapping f of X into the compact space Y which contains only one point not in X leads to the Freudenthal compactification [15]. Another application leads to Carathéodory's construction of prime ends.

The order of the homology group H_g of S_g for a tame connected knot K in S is equal [14] to the resultant of $t^g - 1$ and the Alexander polynomial $\Delta(t)$. To obtain the finer structure of H_g , one has recourse to the algorithm of Seifert [22], which involves finding the g th powers of certain $2h \times 2h$ matrices, where h is the genus of K . Using Fibonacci numbers this can be carried out explicitly¹³ for $h = 1$. For the cloverleaf knot (or for any other knot of genus 1 with the same polynomial $1 - t + t^2$), the homology group H_g depends only on the residue class of $g \bmod 6$; S_g is a Poincaré space for $g \equiv \pm 1 \pmod{6}$. Using p -adic numbers, Artin proved that among the cyclic coverings S_g ($g > 1$) of knots of genus 1 these are the only Poincaré spaces.

The linking invariants of a closed oriented 3-manifold M were defined by Seifert [23]. His definition depends on a particular choice of basis for the torsion group of M . R. C. Blanchfield [4] developed an invariant definition for linking invariants (differing slightly from those of Seifert). Given a matrix F of boundary relations and a corresponding matrix S of intersection numbers these invariants are easily calculated [5] from certain "hybrid" determinants $(F \bowtie S)$ made up of some columns of F and the remaining columns from S . These linking invariants have also been calculated explicitly for the knots of genus one.¹³

In euclidean 3-space the *total curvature* of a closed curve C of class C'' is the integral $\kappa(C) = \int_C |\mathbf{x}''(s)| ds$ ($s =$ arc length). This definition can be extended to arbitrary closed curves by defining, for any closed polygon P , $\kappa(P)$ to be the sum of the exterior angles of P , and, for any closed curve C , $\kappa(C)$ to be the least upper bound of $\kappa(P)$, P ranging over the polygons inscribed in C . J. W. Milnor [17] defines the *curvature* $\kappa(\mathfrak{C})$ of a type $\mathfrak{C}$ of knot to be the greatest lower bound of $\kappa(C)$, C ranging over $\mathfrak{C}$. He proved [17] that (I) $\kappa(\mathfrak{C}) < \infty$ if and only if $\mathfrak{C}$ is tame; (II) for tame $\mathfrak{C}$, $\kappa(\mathfrak{C})/2\pi$ is a positive integer; (III) for tame $\mathfrak{C}$ and $C \in \mathfrak{C}$, $\kappa(C) > \kappa(\mathfrak{C})$; (IV) $\mathfrak{C}$ is unknotted if and only if $\kappa(\mathfrak{C})/2\pi = 1$.¹⁴ Using the number of elementary divisors of the Alexander matrix M , I have shown [13] that there exist types $\mathfrak{C}$ for which $\kappa(\mathfrak{C})/2\pi$ is any prescribed positive integer. Using Morse's critical point theory, Milnor has generalized [16] some of these results to higher dimensions.

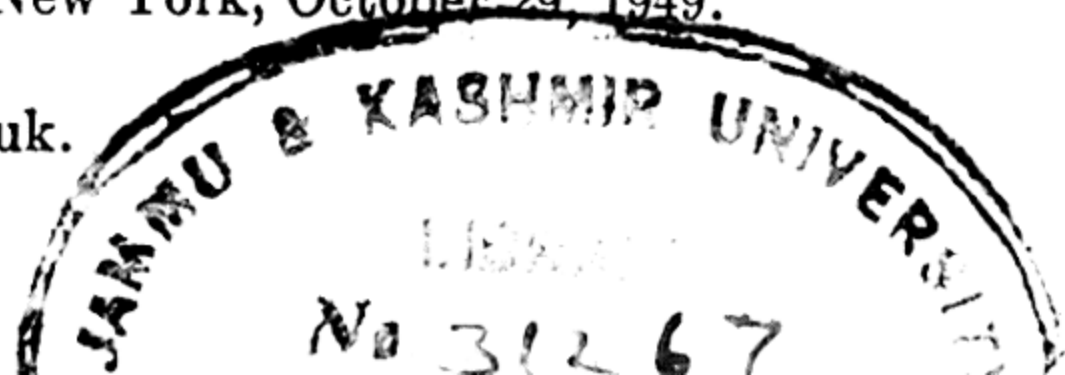
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PRINCETON UNIVERSITY,
PRINCETON, N. J., U. S. A.